M335 MT 5-8

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

STUDIES IN PURE MATHEMATICS

METRIC AND TOPOLOGICAL SPACES UNITS 5-8

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics
Prepared by the Course Team

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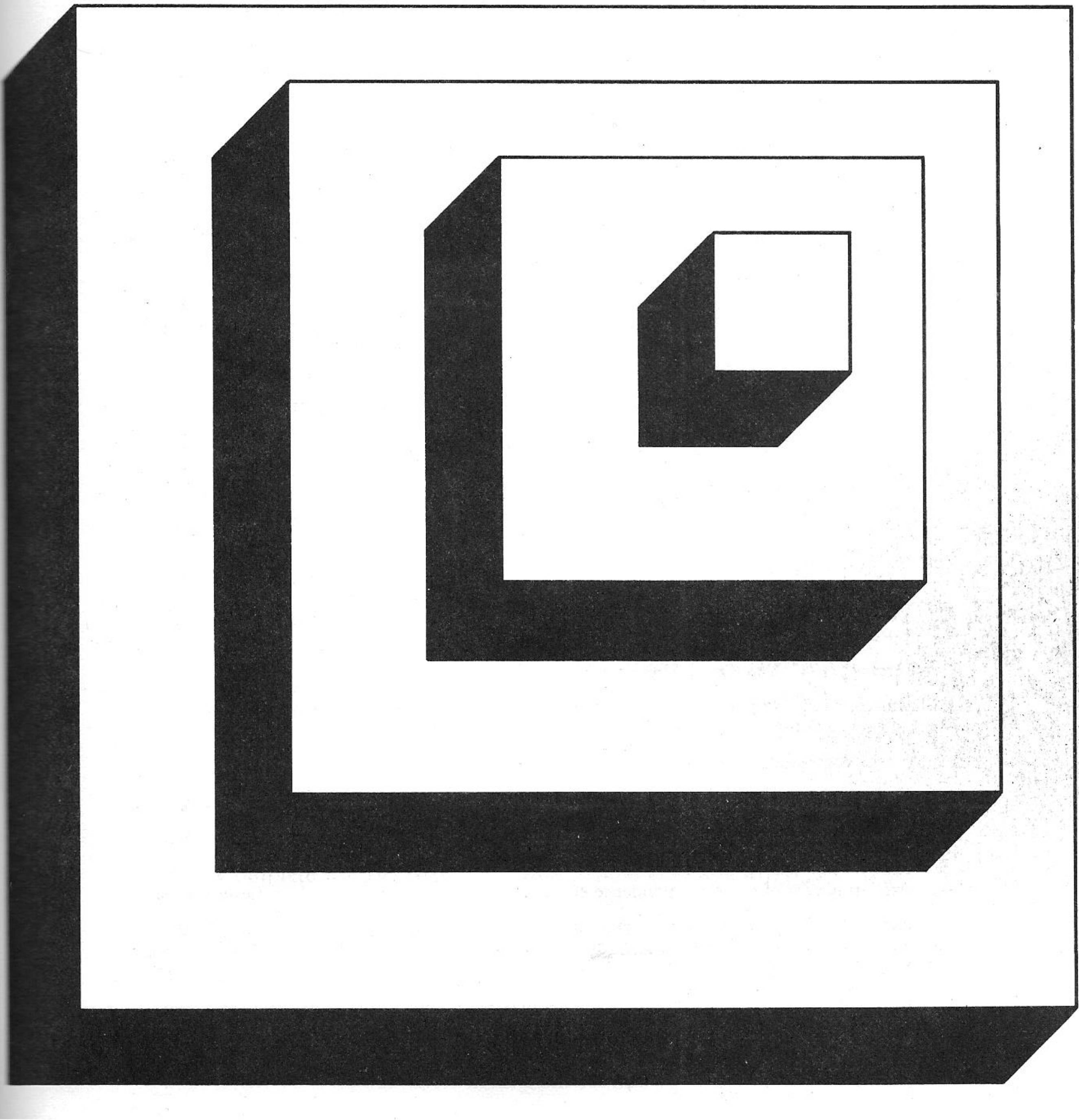

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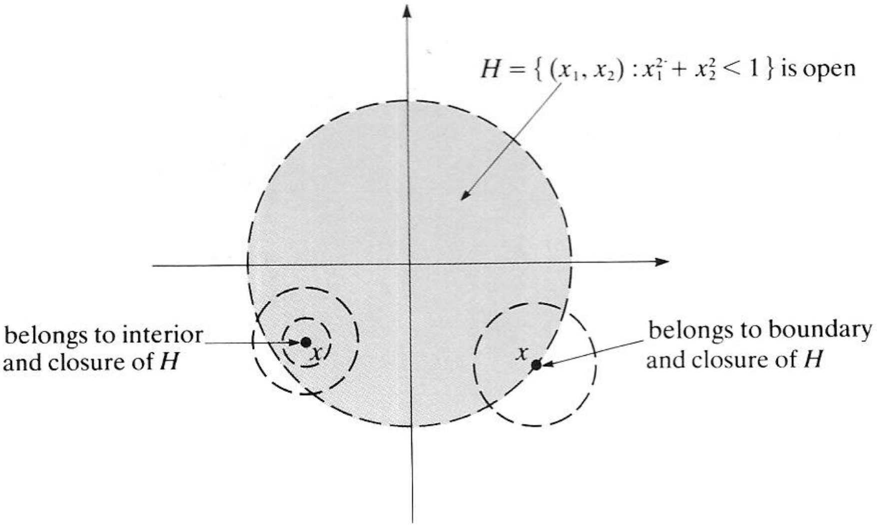
UNIT 5
TOPOLOGICAL SPACES II

5.0 Introduction

In this unit we continue the study of general topological spaces which we began in *Unit 4*. It is based on Chapter 3, Sections 3.6 and 3.7, and Chapter 4 of S.

In Section 5.1, we meet the *closed sets* of a topological space. A subset E of a topological space T is *closed* in T if and only if its complement $T - E$ is open in T. From De Morgan's Laws (Proposition 3 of *Unit 1*) and the properties (T1)–(T3) of open sets, we find that closed sets have properties which are, in a sense, dual to (T1)–(T3).

In Sections 5.2 and 5.3, we study the operations on a subset of a topological space T of forming its *closure, interior* and *boundary*, which are also subsets of T. To give you some intuitive feeling for these concepts, let us consider these operations on a subset H of a metric space M. The *closure* of H consists of all points x of M such that every open ball with centre x contains a point of H. The *interior* of H consists of all points of M such that some open ball with centre x is contained in H. The *boundary* of H consists of all points of M such that every open ball with centre x contains a point of H and a point of the complement of H. These definitions coincide with what your geometric intuition would suggest from consideration of specific subsets H of $\mathbf{R}^2$ or $\mathbf{R}^3$ with their usual metrics (see, for example, the diagram below); however, you will meet other examples of sets and spaces where geometric intuition is no help at all in determining closures, interiors or boundaries.


$H = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$ is open
belongs to interior and closure of H
x
x
belongs to boundary and closure of H

In SAQ 19 of *Unit 3*, we saw that distinct points of a metric space are contained in disjoint open sets. If a topological space T satisfies the condition that distinct points are contained in disjoint open sets, then T is called a *Hausdorff space*. We look at Hausdorff spaces in Section 5.5.

In Section 5.6, the notion of *homeomorphism* is introduced. Topological spaces are *homeomorphic* or *topologically equivalent* if there is a one-one correspondence f between them such that both f and f^{-1} are continuous. In topology, homeomorphic spaces are regarded as essentially the same, in the way that isomorphic groups are regarded as essentially the same in group theory. A property P of topological spaces which is preserved by homeomorphism (in the sense that whenever S and T are homeomorphic spaces, and S has the property P, then T has the property P) is called a *topological property*. We shall see that the property of being a Hausdorff space and the property of being a metrizable space are both topological properties.

5.1 Closed Sets

In this section we look at the collection of subsets of a topological space T which are the complements of open sets of T.

READ S from the beginning of Section 3.7 on page 59, to page 61, line 16.

NOTES (i) *Page 60, Proposition 3.7.4*

A proof of this goes as follows. Since $T = T - \varnothing$ and $\varnothing = T - T$, we see that T and $\varnothing$ are closed sets of T. Let E and F be closed sets of T. By Proposition 3 of *Unit 1*,

$$T - (E \cup F) = (T - E) \cap (T - F).$$

But $T - E$ and $T - F$ are open sets, so that $T - (E \cup F)$, being the intersection of two open sets, is open. Hence $E \cup F$ is a closed set of T. Finally, let $(E_i)_{i \in I}$ be a family of closed sets of T and let $D = \bigcap_{i \in I} E_i$. By Proposition 3 of *Unit 1*,

$$T - D = \bigcup_{i \in I} (T - E_i).$$

But $T - E_i$ is an open set for each i, so that $T - D$ is an open set, by Axiom (T2). Hence D is a closed set of T.

(ii) *Page 61, line 12*

Let us state several useful set-theoretic facts, for future reference.

Let $f\colon T_1 \longrightarrow T_2$ be a function, U a subset of T_1 and V a subset of T_2. Then

$$f(T_1) - f(U) \subset f(T_1 - U)$$

and

$$f^{-1}(T_2 - V) = T_1 - f^{-1}(V).$$

These are simply restatements of the results in SAQ 29 of *Unit 1*, with A, B, C_1, C_2, D_1, D_2 replaced by T_1, T_2, T_1, U, T_2, V, respectively. The next useful fact follows immediately from the definition of a bijection.

Let $f\colon T_1 \longrightarrow T_2$ be a bijection and $V \subset T_1$. Then $T_2 - f(V) = f(T_1 - V)$.

(iii) *Page 60, Examples 3.7.2 and 3.7.3(a)*

The subsets of **R** which are intervals are listed on **S** page xiii. The 'open intervals' are those of the form (a, b), $(-\infty, b)$ or (a, ∞), together with **R** itself. We know that the open intervals are open in **R** with the usual topology. The intervals of the other forms are not open in **R**. For example, $[a, b)$ is not open in **R**, for there exists no $\varepsilon > 0$ such that $B_\varepsilon(a) = (a - \varepsilon, a + \varepsilon) \subset [a, b)$.

The 'closed intervals' are those of the form $[a, b]$, $(-\infty, b]$ or $[a, \infty)$, together with **R** itself. Since

$$\mathbf{R} - \mathbf{R} = \varnothing, \quad \mathbf{R} - [a, b] = (-\infty, a) \cup (b, \infty),$$
$$\mathbf{R} - (-\infty, b] = (b, \infty) \quad \text{and} \quad \mathbf{R} - [a, \infty) = (-\infty, a)$$

we see that the closed intervals are closed in **R**, with the usual topology. The intervals of the other forms are not closed in **R**. For example, $[a, b)$ is not closed in **R**. To see this we note that

$$\mathbf{R} - [a, b) = (-\infty, a) \cup [b, \infty)$$

and $K = (-\infty, a) \cup [b, \infty)$ is not open in **R**, for there exists no $\varepsilon > 0$ such that $B_\varepsilon(b) = (b - \varepsilon, b + \varepsilon) \subset K$. ▲

EXAMPLE 1 In Example 1 of *Unit 4*, we considered the topological spaces $T = \{A, \mathscr{T}\}$ and $T_1 = \{A, \mathscr{T}_1\}$ where $A = \{a, b, c\}$,

$$\mathscr{T} = \{\varnothing, \{a\}, \{b, c\}, A\}$$

and

$$\mathscr{T}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, A\}.$$

The closed sets of T are the complements of the open sets of T. Thus the set of closed sets of T is

$$\{A, \{b, c\}, \{a\}, \varnothing\}.$$

Here we have a surprising situation: a set is closed in T if and only if it is open in T!

The set of closed sets of T_1 is

$$\{A, \{b, c\}, \{a, c\}, \{c\}, \{a\}, \varnothing\}.$$

Remark. The above example shows that a subset of a topological space can be said to be 'unlike a door' in that it may be simultaneously open and closed. Alternatively, as shown by the subset $\{b\}$ of A in Example 1, or the interval $[0, 1)$ of **R**, with the usual topology, it may be neither open nor closed.

EXAMPLE 2 In SAQ 22 of *Unit 4*, we considered the 'left-ray topology' $\mathscr{T}$ on **R** consisting of $\varnothing$, **R** and all unbounded open intervals of the form $(-\infty, a)$, $a \in \mathbf{R}$. In addition to $\varnothing$ and **R**, the closed sets of $T = \{\mathbf{R}, \mathscr{T}\}$ are all sets of the form

$$\mathbf{R} - (-\infty, a) = [a, \infty), \quad a \in \mathbf{R}.$$

Now let $T' = [0, 1]$. By **S** Proposition 3.7.6, the closed sets of the subspace T' of T are all sets of the form $T' \cap W$ where W is closed in T. Thus the closed sets of T' are $\varnothing$, $\{1\} = T' \cap [1, \infty)$ and all closed intervals of the form $[a, 1]$ where $0 \leq a < 1$.

EXAMPLE 3 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the topology on **R** introduced in Example 3 of *Unit 4*. (Recall that V is open in T if and only if either $0 \notin V$ or $\mathbf{R} - V$ is finite.)

The closed sets of T are the sets $\mathbf{R} - V$ where V is open in T. Thus the closed sets of T are those subsets E of **R** such that either $0 \in E$ or E is finite. Thus W is a set which is both open and closed in T if and only if either $0 \in W$ and $\mathbf{R} - W$ is finite or W is a finite set not containing 0.

SAQ 1 Find the set of all closed sets of a topological space $\{A, \mathscr{T}\}$ if: (i) $\mathscr{T}$ is the discrete topology (**S** Example 3.1.4); (ii) $\mathscr{T}$ is the indiscrete topology (**S** Example 3.1.6).

SAQ 2 Let $T = \{A, \mathscr{T}\}$ where $A = \{a, b, c, d, e\}$ and

$$\mathscr{T} = \{\varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, A\}.$$

(a) Find (i) a non-empty proper subset of A which is both open in T and closed in T, and (ii) a subset of A which is neither open in T nor closed in T.

(b) Let $H = \{c, d, e\}$. Find the closed sets of the subspace H of T.

SAQ 3 Prove that a subset E of a topological space T is a closed set of T if and only if for each point x of $T - E$ there exists an open set V_x of T such that $x \in V_x$ and $V_x \cap E = \varnothing$.

SAQ 4 Let E_1, E_2 be closed sets of the topological spaces T_1, T_2 respectively. Prove that $E_1 \times E_2$ is a closed set of the topological product $T_1 \times T_2$. [Hint: Use SAQ 3.]

SAQ 5 Let $\mathscr{T}_1, \mathscr{T}_2$ be topologies for a set A such that $\mathscr{T}_1 \subset \mathscr{T}_2$. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be the sets of closed sets of the topological spaces $\{A, \mathscr{T}_1\}$ and $\{A, \mathscr{T}_2\}$ respectively. Does it follow that $\mathscr{E}_1 \subset \mathscr{E}_2$ or that $\mathscr{E}_2 \subset \mathscr{E}_1$? (Justify your answer.)

SAQ 6 (a) Let T be a topological space and let $f: T \longrightarrow \mathbf{R}$ be a continuous function where **R** has its usual topology. Prove that for each real number c the set

$$\{x \in T : f(x) \leq c\}$$

is closed in T. [Hint: Use **S** Proposition 3.7.8.]

(b) Deduce that, if $M = \{A, d\}$ is a metric space, $a \in A$ and $c \in \mathbf{R}_+$, then the set

$$\{x \in A : d(a,x) \leq c\}$$

is closed in M. [Hint: Use SAQ 11 of *Unit* 3.]

SOLUTION TO SAQ 1 (i) If $\mathscr{T}$ is the discrete topology, every subset of A is open. Hence, for any set $V \in 2^A$, $A - V$ is an open set; so that V is a closed set. Thus every subset of A is a closed set.

(ii) If $\mathscr{T}$ is the indiscrete topology then $\varnothing$ and A are the only open sets, so that A and $\varnothing$ are the only closed sets. ■

SOLUTION TO SAQ 2 (a) The set of closed sets of T is

$$\{A, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}, \varnothing\}.$$

(i) Then $\{a\}$ is an example of a non-empty proper subset which is an open set and a closed set of T. ($\{b, c, d, e\}$ is another.)

(ii) $\{b\}$ is a subset of A which neither open in T nor closed in T. (There are *many* others!)

(b) By S Proposition 3.7.6, the closed sets of H are H, $\{e\}$, $\varnothing$. ■

SOLUTION TO SAQ 3 A subset E of a topological space T is closed in T if and only if $T - E$ is open in T. But by SAQ 4 of *Unit 4*, $T - E$ is open in T if and only if for each x in $T - E$ there exists V_x open in T such that $x \in V_x \subset T - E$. Since $V_x \subset T - E$ means that $V_x \cap E = \varnothing$, this completes the solution. ■

SOLUTION TO SAQ 4 We shall use the result of SAQ 3. Let $x = (x_1, x_2)$ be a point of $T_1 \times T_2$ such that $x \notin E_1 \times E_2$. Then either $x_1 \notin E_1$ or $x_2 \notin E_2$. If $x_1 \notin E_1$ then since E_1 is a closed set of T_1, by SAQ 3 there exists an open set W of T_1 such that $x_1 \in W$ and $W \cap E_1 = \varnothing$. If $V_x = W \times T_2$ then V_x is open in $T_1 \times T_2$, $x \in V_x$ and $V_x \cap (E_1 \times E_2) = \varnothing$.

Similarly, if $x_2 \notin E_2$, we can find an open set of V_x of $T_1 \times T_2$ such that $x \in V_x$ and $V_x \cap (E_1 \times E_2) = \varnothing$. It follows from SAQ 3 that $E_1 \times E_2$ is a closed set of $T_1 \times T_2$. ■

SOLUTION TO SAQ 5 If $E \in \mathscr{E}_1$ then $A - E \in \mathscr{T}_1$ so that $A - E \in \mathscr{T}_2$ and hence $A - (A - E) = E \in \mathscr{E}_2$. Thus $\mathscr{T}_1 \subset \mathscr{T}_2 \Rightarrow \mathscr{E}_1 \subset \mathscr{E}_2$. ■

SOLUTION TO SAQ 6 (a) The set $(-\infty, c]$ is closed in $\mathbf{R}$. Thus by S Proposition 3.7.8,

$$f^{-1}((-\infty, c]) = \{x \in T : f(x) \leq c\}$$

is closed in T.

(b) By SAQ 11 of *Unit 3*, the map $A \longrightarrow \mathbf{R}$ given by $x \longmapsto d(a,x)$, $x \in A$, is continuous. It follows from part (a) that $\{x \in A : d(a,x) \leq c\}$ is closed in M. ■

5.2 Limit Points and Closure

READ S from page 61, line 17 to page 64, line 8, omitting Definition 3.7.20 and Example 3.7.21.

NOTES (i) *Page 61, Definition 3.7.9*
In M332, *Complex Analysis*, a limit point is called a *cluster point*.

(ii) *Page 61, Proposition 3.7.12*
It is clear that if $x \in Cl(H)$ then every open set containing x contains a point of H. Conversely, let x be a point of T with the property that every open set containing x contains a point of H. If $x \in H$ then $x \in Cl(H)$. If $x \notin H$ then every open set containing x contains some point of H (necessarily distinct from x) so that x is a limit point of H and again $x \in Cl(H)$. This completes the proof of Proposition 3.7.12.

(iii) *Page 62, Proposition 3.7.15*
In the 1975 edition of **S**, the statement of part (b) should read

'$H \subset K \Rightarrow Cl(H) \subset Cl(K)$'.

(iv) *Page 62, Proposition 3.7.16*
To say that $Cl(H)$ is the *smallest closed set containing* H means that $Cl(H)$ is a closed set containing H and that it is contained in any other closed set that contains H.

(v) *Page 63, line 8*
For a simple example showing that equality does not necessarily hold in Proposition 3.7.17, let $H_1 = (-\infty, 0)$ and $H_2 = (0, \infty)$ in **R**, with the usual topology. Then $H_1 \cap H_2 = \varnothing$ so that $Cl\,(H_1 \cap H_2) = \varnothing$. But $Cl\,(H_1) = (-\infty, 0]$ and $Cl\,(H_2) = [0, \infty)$ so that $Cl\,(H_1) \cap Cl\,(H_2) = \{0\}$.

A more striking example is as follows. Let $H_1 = \mathbf{Q}$, $H_2 = \mathbf{R} - \mathbf{Q}$ in **R** (with the usual topology). Then $Cl(H_1) = \mathbf{R}$, $Cl(H_2) = \mathbf{R}$ (by the rational and irrational density properties) so $Cl(H_1) \cap Cl(H_2) = \mathbf{R}$, but $Cl(H_1 \cap H_2) = Cl(\varnothing) = \varnothing$.

(vi) *Page 63, line −13*
S is pointing out here that if $(H_i)_{i \in I}$ is a family of subsets of a topological space, $H = \bigcup\limits_{i \in I} H_i$ and $K = \bigcup\limits_{i \in I} Cl(H_i)$, then it is not true in general that $Cl(H) \subset K$. **S** Example 3.7.19 provides an appropriate counter-example. Of course, $H_i \subset H$ for each i, so that $Cl(H_i) \subset Cl(H)$, and hence $K \subset Cl(H)$.

(vii) *Page 64, Proposition 3.7.23*
An alternative proof is as follows. The assertion is that

$$Cl(B_\varepsilon(x)) \subset \{y : d(x,y) \leq \varepsilon\}.$$

This is an immediate consequence of the result of SAQ 6(b), for $\{y : d(x,y) \leq \varepsilon\}$ is a closed set which contains $B_\varepsilon(x)$. ▲

EXAMPLE 4 For the topological space T studied in Example 1, the non-closed sets are $\{b\}$, $\{c\}$, $\{a,b\}$ and $\{a,c\}$. Since the closure of a set is the smallest closed set in which it is contained, we see that

$$Cl(\{b\}) = Cl(\{c\}) = \{b,c\}$$

and that

$$Cl(\{a,b\}) = Cl(\{a,c\}) = A.$$

Also, for the topological space T_1 considered in Example 1, the non-closed sets are $\{b\}$ and $\{a,b\}$; we see that

$$Cl(\{b\}) = \{b,c\} \quad \text{and} \quad Cl_s(\{a,b\}) = A.$$

EXAMPLE 5 Let us find the set of limit points of the subset

$$H = \mathbf{Q} \cap [0, \sqrt{2})$$

of **R**, with the usual topology.

As H consists of those rational numbers in the interval $[0, \sqrt{2})$, it is not surprising that the rational density property will be one of our key tools here. We shall show that limit points of H necessarily belong to the closed interval $[0, \sqrt{2}]$, and then that every point in $[0, \sqrt{2}]$ is actually a limit point of H.

If $b \in \mathbf{R}$ and $b > \sqrt{2}$, then $(\sqrt{2}, \infty)$ is an open set of **R** which contains b and is disjoint from H, from which it follows that b is not a limit point of H. If $b \in \mathbf{R}$ and $b < 0$, then $(-\infty, 0)$ is an open set of **R** which contains b and is disjoint from H, and again we see that b is not a limit point of H. Thus if b is a limit point of H then $0 \leq b \leq \sqrt{2}$.

Next, we shall show that each point of $[0, \sqrt{2}]$ is a limit point of H. As the arguments are slightly different in the two cases, we shall consider the sets $[0, \sqrt{2})$ and $\{\sqrt{2}\}$ separately.

Let $\varepsilon > 0$ be given. If $0 \leq b < \sqrt{2}$, then $b < \operatorname{Min}\{\sqrt{2}, b + \varepsilon\}$. Thus, by the rational density property (*Unit 2*, Section 2.1) there exists a rational number r such that

$$b < r < \operatorname{Min}\{\sqrt{2}, b + \varepsilon\}.$$

Since $r \in H$, we see that $(b - \varepsilon, b + \varepsilon)$ contains a point of H distinct from b. Hence b is a limit point of H. Also, $\sqrt{2}$ is a limit point of H, for if $\varepsilon > 0$ then, by the rational density property again, there exists a rational number s such that

$$\sqrt{2} > s > \operatorname{Max}\{0, \sqrt{2} - \varepsilon\}.$$

Then $s \in H$, so that $(\sqrt{2} - \varepsilon, \sqrt{2} + \varepsilon)$ contains a point of H. Therefore $\sqrt{2}$ is a limit point of H. Hence the set of limit points of H in $\mathbf{R}$ is $[0, \sqrt{2}]$. It follows that in $\mathbf{R}$ (with the usual topology),

$$Cl(H) = [0, \sqrt{2}].$$

Notice that, since $\sqrt{2}$ is an irrational number, $H = \mathbf{Q} \cap [0, \sqrt{2}]$, so that H is a closed set of the subspace $\mathbf{Q}$ of $\mathbf{R}$ by S Proposition 3.7.6. Thus $Cl(H) = H$ in the subspace $\mathbf{Q}$.

AMPLE 6 Let us consider the topological space T, with underlying set $\mathbf{R}$, specified and studied in Example 3. Let H be a subset of $\mathbf{R}$ which is not closed in T. Then $0 \notin H$ and H is an infinite set. Let us find the limit points of H in the topological space T. If $x \in \mathbf{R}$ and $x \neq 0$ then $\{x\}$ is an open set of T, and hence x is not a limit point of H. In particular, no point of H is a limit point of H.

However, 0 is a limit point of H. For if V is an open set of T such that $0 \in V$, then $R - V$ is a finite set. Hence $V \cap H \neq \varnothing$, since H is an infinite set, and so 0 is a limit point of H. It follows that $Cl(H) = H \cup \{0\}$ for each non-closed set H of T.

AMPLE 7 Let $M = \{A, d\}$ be a metric space, H a non-empty subset of A and $x \in A$. Since $d(x, y) \geq 0$ for all y, the non-empty set $\{d(x, y) : y \in H\}$ of real numbers is bounded below. We define the *distance of x from H* as

$$d(x, H) = \inf\{d(x, y) : y \in H\}.$$

Plainly, $d(x, H) = 0$ if $x \in H$. The converse result (namely, that if $d(x, H) = 0$ then $x \in H$) is not necessarily true; for example, if $M = \mathbf{R}$ with the usual topology, $H = (0, 1)$ and $x = 1$, then $d(x, H) = 0$ although $x \notin H$.

Next, if $d(x, H) > 0$, then x is not a limit point of H in M. For if $d(x, H) = \delta > 0$, then $d(x, y) \geq \delta$ for all $y \in H$ so that the open ball $B_\delta(x)$ is disjoint from H. Finally, if $x \notin H$ and $d(x, H) = 0$ then x is a limit point of H. To see this, let $\varepsilon > 0$ be given. Since $d(x, H) = 0$, there exists y in H such that $d(x, y) < \varepsilon$ and so $B_\varepsilon(x) \cap H \neq \varnothing$. Since $x \notin H$ it follows that x is a limit point of H. Putting all this together, we can see that

$$Cl(H) = \{x \in A : d(x, H) = 0\}.$$

In particular, *H is a closed set of M if and only if $d(x, H) = 0$ implies $x \in H$*, for then

$$H = \{x \in A : d(x, H) = 0\}.$$

SAQ 7 Let T be an indiscrete space and H be a non-empty subset of T.

(i) What is the closure of H?

(ii) Find the set of limit points of H.

SAQ 8 Let $T = \{\mathbf{R}, \mathscr{T}\}$, where $\mathscr{T}$ is the left-ray topology for $\mathbf{R}$ considered in Example 2, and let H be a non-empty subset of T. Find $Cl(H)$. [Hint: You should consider separately the cases in which H is bounded below and H is not bounded below.]

***SAQ 9** Let T_1, T_2 be topological spaces and $f\colon T_1 \longrightarrow T_2$ be a function.

(a) Prove that f is continuous if and only if

$$Cl(f^{-1}(K)) \subset f^{-1}(Cl(K)) \text{ for all } K \subset T_2.$$

(b) Prove that f is continuous if and only if

$$f(Cl(H)) \subset Cl(f(H)) \text{ for all } H \subset T_1.$$

(Remember that to prove a statement of the variety 'A holds if and only if B holds', it is often best to prove first that A implies B and then that B implies A.)

SAQ 10 Suppose that $H \subset T_1 \subset T_2$ where T_1 is a subspace of the topological space T_2, and let K_i denote the closure of H in T_i for $i = 1,2$. Prove that $K_1 = T_1 \cap K_2$.

SAQ 11 Prove that if $H \subset T_1$ and $K \subset T_2$ where T_1 and T_2 are topological spaces, then

$$Cl(H \times K) = Cl(H) \times Cl(K),$$

where (obviously enough) $Cl(H \times K)$ means the closure of $H \times K$ in the topological product space $T_1 \times T_2$.

SOLUTION TO SAQ 7

(i) The only closed sets are $\varnothing, T$; since $H \neq \varnothing$ we must therefore have $Cl(H) = T$.

(ii) It follows from (i) that every point of $T - H$ is a limit point of H. To see which points of H are also limit points of H, let $x \in H$. Then T is the only open set containing x; hence x is a limit point if and only if

$$T \cap (H - \{x\}) \neq \varnothing,$$

which holds provided $H \neq \{x\}$. Thus every point of H is a limit point of H provided H contains more than one point. If $H = \{x\}$, then x is not a limit point of H. Thus the set of limit points of H is equal to T if H has more than one point and is equal to $T - \{x\}$ if H is the singleton set $\{x\}$. ■

SOLUTION TO SAQ 8 We recall that the non-empty proper closed sets of T are the unbounded closed intervals $[a, \infty)$, where $a \in \mathbf{R}$. If H is not bounded below, then for each real number a there exists x in H such that $x < a$. Thus no proper closed set of T contains H. Hence if H is not bounded below then $Cl(H) = \mathbf{R}$.

Now suppose that H is bounded below and let $b = \inf H$. Then $x \geq b$ for all $x \in H$ so that $H \subset [b, \infty)$. Also, if $H \subset [a, \infty)$ then a is a lower bound for H so that $b \geq a$ and hence $[b, \infty) \subset [a, \infty)$. Since the closure of H is the smallest closed set which contains H, we see that $Cl(H) = [b, \infty)$. ■

SOLUTION TO SAQ 9

(a) Let $f\colon T_1 \longrightarrow T_2$ be continuous and K a subset of T_2. Since $K \subset Cl(K)$, it follows that $f^{-1}(K) \subset f^{-1}(Cl(K))$. Now, by **S** Proposition 3.7.8, $f^{-1}(Cl(K))$ is a closed set of T_1. It is a closed set that contains $f^{-1}(K)$; and so, since $Cl(f^{-1}(K))$ is the smallest closed set that contains $f^{-1}(K)$ (as discussed in Note (iv)), it follows that $Cl(f^{-1}(K)) \subset f^{-1}(Cl(K))$.

Conversely, suppose that $Cl(f^{-1}(K)) \subset f^{-1}(Cl(K))$ for all $K \subset T_2$. In particular, if K is a closed set of T_2 then $Cl(K) = K$ so that $Cl(f^{-1}(K)) \subset f^{-1}(K)$; from which it follows that $Cl(f^{-1}(K)) = f^{-1}(K)$ by **S** Proposition 3.7.16. Thus $f^{-1}(K)$ is a closed set of T_1 for each closed set K of T_2 so that, by **S** Proposition 3.7.8, f is continuous.

(b) Suppose that $f(Cl(H)) \subset Cl(f(H))$ for all $H \subset T_1$. In particular, if we take $H = f^{-1}(K)$ where $K \subset T_2$ then

$$f(Cl(f^{-1}(K))) \subset Cl(f(f^{-1}(K))) \subset Cl(K).$$

From Proposition 4 of *Unit 1*, we see that

$$Cl(f^{-1}(K)) \subset f^{-1}(Cl(K)),$$

and hence f is continuous by the result of part (a).

Now let f be continuous and let H be a subset of T_1. If $b \in f(Cl(H))$ then $b = f(a)$ where $a \in Cl(H)$. We need to show that b is in the closure of $f(H)$;

that is, that every open set V containing b meets $f(H)$ (by **S** Proposition 3.7.12). So, let V be an open set of T_2 such that $b \in V$. Then $f^{-1}(V)$ is an open set of T_1 and $a \in f^{-1}(V)$. Since $a \in Cl(H)$, it follows from **S** Proposition 3.7.12 that $f^{-1}(V) \cap H \neq \varnothing$. If $x \in f^{-1}(V) \cap H$ then $f(x) \in V \cap f(H)$. Thus $V \cap f(H) \neq \varnothing$. From **S** Proposition 3.7.12, we see that $b \in Cl(f(H))$. Hence $f(Cl(H)) \subset Cl(f(H))$. ■.

LUTION SAQ 10 Since K_1 is a closed set of T_1, it follows from **S** Proposition 3.7.6 that there exists a closed set E of T_2 such that $K_1 = T_1 \cap E$. Since $H \subset K_1 \subset E$ and K_2 is the smallest closed set of T_2 which contains H, we see that $K_2 \subset E$. Thus $T_1 \cap K_2 \subset T_1 \cap E$, so that $T_1 \cap K_2 \subset K_1$. But $H \subset T_1 \cap K_2$ and, by **S** Proposition 3.7.6, $T_1 \cap K_2$ is a closed set of T_1. Since K_1 is the smallest closed set of T_1 which contains H, it follows that $K_1 \subset T_1 \cap K_2$. Thus $K_1 = T_1 \cap K_2$. ■

TION SAQ 11 Since $H \times K \subset Cl(H) \times Cl(K)$ and $Cl(H) \times Cl(K)$ is a closed set of $T_1 \times T_2$ (by SAQ 4), it follows that $Cl(H \times K) \subset Cl(H) \times Cl(K)$. Now suppose that $(x, y) \in Cl(H) \times Cl(K)$, and let U be an open set of $T_1 \times T_2$ such that $(x, y) \in U$. We need to show that U meets $H \times K$. By **S** Proposition 3.5.2, there exist open sets V, W of T_1, T_2 respectively, such that $(x, y) \in V \times W \subset U$. Since $x \in Cl(H)$ and V is an open set of T_1, it follows from **S** Proposition 3.7.12 that $V \cap H \neq \varnothing$. Similarly, $W \cap K \neq \varnothing$. Therefore $(V \times W) \cap (H \times K) \neq \varnothing$ and so $U \cap (H \times K) \neq \varnothing$. Hence $(x, y) \in Cl(H \times K)$. Thus

$$Cl(H \times K) = Cl(H) \times Cl(K). \quad ■$$

5.3 Interiors and Boundaries

In this section we consider two more operations on subsets of a topological space, and examine their connections with the closure operation introduced in the previous section.

READ S Page 64, lines 9 to 15.

Propositions 3.7.15 and 3.7.16 have analogues for interiors.

SITION 1

(a) H is open in T if and only if $H = \operatorname{Int} H$.

(b) $H \subset K \Rightarrow \operatorname{Int} H \subset \operatorname{Int} K$.

(c) $\operatorname{Int}(\operatorname{Int} H) = \operatorname{Int} H$.

(d) $\operatorname{Int} H$ is open.

PROOF It is convenient to prove the parts in a slightly different order from that in which they are stated.

(d) Since $\operatorname{Int} H$ is the union of open sets of T, it must be itself open (by Axiom (T3)).

(a) It is always the case that $\operatorname{Int} H \subset H$.

So suppose first that H is open. Then H is one of the sets whose union is $\operatorname{Int} H$, so that $H \subset \operatorname{Int} H$. Combining this with the previous inclusion, we deduce that $H = \operatorname{Int} H$.

Conversely, if $H = \operatorname{Int} H$, H itself must be open since $\operatorname{Int} H$ is open.

(c) This follows at once from (a) and (d).

(b) If $H \subset K$, then every subset of H that is open in T is also a subset of K that is open in T. Hence $\operatorname{Int} K$ is the union of (i) all subsets of H that are open in T, and (ii) possibly other sets as well. It follows that $\operatorname{Int} K \supset \operatorname{Int} H$, as required. ■

SITION 2 $\operatorname{Int} H$ is the largest open subset of H.

PROOF By Proposition 1, $\operatorname{Int} H$ is an open subset of H. So suppose that V is an open subset of H with $\operatorname{Int} H \subset V$. Since V is one of the sets whose union is $\operatorname{Int} H$, by definition, it follows that $V \subset \operatorname{Int} H$. Combining these two inclusions, we see that $V = \operatorname{Int} H$. It follows that there is no larger open subset of H. ■

READ S Page 64, Definition 3.7.31 and Example 3.7.32.

EXAMPLE 8 In $\mathbf{R}^2$ with the usual topology, the boundary of the disc

$$D_a = \{(x_1, x_2): x_1^2 + x_2^2 < a^2\}$$

is the circumference $\{(x_1, x_2): x_1^2 + x_2^2 = a^2\}$. □

There is no complete analogue of Proposition 3.7.15 for boundaries. For example, it is not necessarily true that, if $H \subset K$, then $b(H) \subset b(K)$. For instance, using the notation of Example 8, $D_1 \subset D_2$ but

$$b(D_1) = \{(x_1, x_2): x_1^2 + x_2^2 = 1\}$$

is not a subset of

$$b(D_2) = \{(x_1, x_2): x_1^2 + x_2^2 = 4\}.$$

Also, it is not necessarily true that $b(b(H)) = b(H)$. For example, if $H = \mathbf{Q} \cap [0,1]$, then, in $\mathbf{R}$ with the usual topology, $b(H) = [0,1]$ so that $b(b(H)) = \{0,1\}$.

PROPOSITION 3 (a) $b(H)$ is closed.

(b) $b(H) = b(T - H)$.

PROOF (a) By definition, $b(H)$ is the intersection of two closed sets, so itself is closed, by S Proposition 3.7.4.

(b) Since $T - (T - H) = H$, we have

$$\begin{aligned} b(T - H) &= Cl(T - H) \cap Cl(T - (T - H)) \\ &= Cl(T - H) \cap Cl(H) \\ &= b(H). \quad ■ \end{aligned}$$

We now prove several additional results relating closure, interior and boundary.

PROPOSITION 4 If H is a subset of a topological space T, then

$$Cl(T - H) = T - \operatorname{Int} H.$$

PROOF Since $\operatorname{Int} H \subset H$, it follows that

$$T - H \subset T - \operatorname{Int} H.$$

But $T - \operatorname{Int} H$ is a closed set of T, and so by S Proposition 3.7.15(b) it follows that

$$Cl(T - H) \subset T - \operatorname{Int} H.$$

We next prove the reverse inclusion

$$T - \operatorname{Int} H \subset Cl(T - H).$$

Since $Cl(T - H) \supset T - H$ we have

$$T - Cl(T - H) \subset T - (T - H) = H;$$

but $T - Cl(T - H)$ is open so, by definition of Int H,

$$T - Cl(T - H) \subset \operatorname{Int} H.$$

Hence

$$T - \operatorname{Int} H \subset Cl(T - H).$$

Therefore

$$T - \operatorname{Int} H = Cl(T - H). \quad ■$$

OPOSITION 5 If H is a subset of a topological space T then

(i) $\operatorname{Int} H$ and $b(H)$ are disjoint;

(ii) $(\operatorname{Int} H) \cup b(H) = Cl(H)$;

(iii) $b(H) = Cl(H) - \operatorname{Int} H$.

PROOF (i) Since $b(H) \subset Cl(T - H)$ by definition, and $\operatorname{Int} H$ and $Cl(T - H)$ are disjoint by Proposition 4, we see that $\operatorname{Int} H$ and $b(H)$ are disjoint.

(ii) Since $b(H) \subset Cl(H)$ and $\operatorname{Int} H \subset H \subset Cl(H)$, it follows that

$$(\operatorname{Int} H) \cup b(H) \subset Cl(H).$$

In the opposite direction, if $x \in Cl(H)$, then since $b(H) = Cl(H) \cap Cl(T - H)$, either $x \in b(H)$ or $x \notin Cl(T - H)$. But if $x \notin Cl(T - H)$ then $x \in \operatorname{Int} H$ by Proposition 4. Thus

$$(\operatorname{Int} H) \cup b(H) = Cl(H).$$

(iii) This follows immediately from (i) and (ii). ■

OPOSITION 6 Let H be a subset of a topological space T. Then

(i) H is closed if and only if $b(H) \subset H$; and

(ii) H is open if and only if $b(H) \cap H = \varnothing$.

In other words, a set is closed if and only if it contains all its boundary and open if and only if it contains none of its boundary.

PROOF (i) Since $b(H) \subset Cl(H)$, if H is closed then $b(H) \subset H$. Conversely, if $b(H) \subset H$ then, since $\operatorname{Int} H \subset H$, it follows from Proposition 5 that $Cl(H) \subset H$. Therefore $Cl(H) = H$ and so H is a closed set.

(ii) If H is an open set, then $\operatorname{Int} H = H$ and so $H \cap b(H) = \varnothing$, by Proposition 5. Also, $H \subset (\operatorname{Int} H) \cup b(H)$ by Proposition 5, so that if $H \cap b(H) = \varnothing$ then $H \subset \operatorname{Int} H$ and so $H = \operatorname{Int} H$. Therefore if $H \cap b(H) = \varnothing$, then H is an open set. ■

ROPOSITION 7 A subset H of a topological space T is both open and closed in T if and only if $b(H) = \varnothing$.

PROOF Since H is both open and closed in T if and only if

$$\operatorname{Int} H = H = Cl(H),$$

it follows from Proposition 5 that H is both open and closed if and only if $b(H) = \varnothing$. ■

EXAMPLE 9 For the topological space T, with underlying set $\{a, b, c\}$, introduced in Example 1, the sets $\varnothing$, $\{a\}$, $\{b, c\}$ and A are both open and closed. We saw in Example 4 that

$$Cl(\{b\}) = Cl(\{c\}) = \{b, c\}.$$

Since $\{b\}$ contains no non-empty open set of T, we see that $\operatorname{Int}\{b\} = \varnothing$. Similarly, $\operatorname{Int}\{c\} = \varnothing$. Therefore

$$b(\{b\}) = b(\{c\}) = \{b, c\}.$$

Also from Example 4,

$$Cl(\{a, b\}) = Cl(\{a, c\}) = A.$$

Since $\{a, b\}$ contains the open set $\{a\}$ and no other non-empty open set we see that $\operatorname{Int}\{a, b\} = \{a\}$. Similarly, $\operatorname{Int}\{a, c\} = \{a\}$. Therefore, by Proposition 5(iii),

$$b(\{a, b\}) = b(\{a, c\}) = \{b, c\}.$$

EXAMPLE 10 Let $T = \{\mathbf{R}, \mathscr{T}\}$ be the topological space considered in Examples 3 and 6, and let H be a subset of $\mathbf{R}$. We shall find the interior, closure and boundary of H.

For convenience, we shall deal separately with the cases where $0 \notin H$ and $0 \in H$.

(i) Let $0 \notin H$. Then H is an open set of T. If H is finite then H is also a closed set of T, so that

$$\operatorname{Int} H = H = Cl(H) \quad \text{and} \quad b(H) = \varnothing.$$

If H is infinite then, as shown in Example 6,

$$Cl(H) = H \cup \{0\}.$$

Since H is open, $\operatorname{Int} H = H$ and so $b(H) = \{0\}$ by Proposition 5(iii).

(ii) Let $0 \in H$. Then H is a closed set of T. If $\mathbf{R} - H$ is finite then H is also an open set of T, so that

$$\operatorname{Int} H = H = Cl(H) \quad \text{and} \quad b(H) = \varnothing.$$

If $\mathbf{R} - H$ is infinite then $\mathbf{R} - H$ is not closed, so that, as shown in Example 6, $Cl(\mathbf{R} - H) = (\mathbf{R} - H) \cup \{0\}$. But, by Proposition 4,

$$\mathbf{R} - \operatorname{Int} H = Cl(\mathbf{R} - H) = (\mathbf{R} - H) \cup \{0\}.$$

Therefore

$$\begin{aligned}\operatorname{Int} H &= \mathbf{R} - ((\mathbf{R} - H) \cup \{0\}) = H \cap (\mathbf{R} - \{0\}) \\ &= H - \{0\}.\end{aligned}$$

Since $Cl(H) = H$, it follows from Proposition 2 that $b(H) = \{0\}$.

SAQ 12 Find the interior, closure and boundary of every subset of a topological space $\{A, \mathscr{T}\}$ if (i) $\mathscr{T}$ is the discrete topology, and (ii) $\mathscr{T}$ is the indiscrete topology.

SAQ 13 Since $H \subset Cl(H)$, it follows that

$$\operatorname{Int} H \subset \operatorname{Int}(Cl(H)).$$

By means of a specific counter-example, show that it is not necessarily true that

$$\operatorname{Int} H = \operatorname{Int}(Cl(H)).$$

[Hint: Choose a suitable H in the space T of Example 9.]

SAQ 14 Find the interior, closure and boundary of the following subsets of $\mathbf{R}$ (with the usual topology):

(i) $[0,\frac{1}{2}) \cup (\frac{1}{2}, 1]$; (ii) $[0,\frac{1}{2}) \cup (\frac{1}{2}, 1] \cup \{2\}$; (iii) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$; (iv) $\mathbf{Z}$.

SAQ 15 Let $\mathbf{R}$ have the left-ray topology introduced in Example 2. Find the interior, closure and boundary of the following subsets of $\mathbf{R}$:

(i) $(-\infty, 0]$; (ii) $[0, 1]$; (iii) $[1, \infty)$; (iv) $\mathbf{Z}$.

SAQ 16 Let H be a subset of a metric space $\{A, d\}$. Prove that

$$\operatorname{Int} H = \{x \in A : d(x, A - H) > 0\}$$

and

$$b(H) = \{x \in A : d(x, H) = 0 \text{ and } d(x, A - H) = 0\}.$$

SOLUTION TO SAQ 12

(i) If $T = \{A, \mathscr{T}\}$ where $\mathscr{T}$ is the discrete topology and $H \subset A$, then H is both open and closed in T so that

$$\operatorname{Int} H = H = Cl(H) \quad \text{and} \quad b(H) = \varnothing.$$

(ii) If $T = \{A, \mathscr{T}\}$ where $\mathscr{T}$ is the indiscrete topology and H is a non-empty proper subset of A, then H contains no non-empty open set of T so that

$$\operatorname{Int} H = \varnothing \quad \text{and} \quad Cl(H) = b(H) = A.$$

The only other possibility is that H is either $\varnothing$ or A. In both cases, H is both open and closed in T, so that (as in (i))

$$\operatorname{Int} H = H = Cl(H) \quad \text{and} \quad b(H) = \varnothing. \quad ■$$

SOLUTION TO SAQ 13 In the space T of Example 9, take $H = \{b\}$. Then $\text{Int}\, H = \varnothing$. But $Cl(H) = \{b, c\}$, so that $\text{Int}(Cl(H)) = \{b, c\}$. ■

Remark. There are many available counter-examples; here is another, for good measure.

Let T be $\mathbf{R}$ with the usual topology, and let $H = \mathbf{Q}$. Then $Cl(H) = \mathbf{R}$ and $\text{Int}\,(Cl(H)) = \mathbf{R}$, although $\text{Int}\, H = \varnothing$ so that $Cl(\text{Int}\, H) = \varnothing$.

SOLUTION TO SAQ 14

(i) By direct application of the definitions, it follows that if $H = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ then $\text{Int}\, H = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $Cl(H) = [0, 1]$, so that $b(H) = \{0, \frac{1}{2}, 1\}$.

(ii) If $H = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \cup \{2\}$ then $\text{Int}\, H = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $Cl(H) = [0, 1] \cup \{2\}$ so that $b(H) = \{0, \frac{1}{2}, 1, 2\}$.

(Notice particularly that $2 \notin \text{Int}\, H$, since no open subset of H contains the point 2.)

(iii) If $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ then $\text{Int}\, H = \varnothing$ and $Cl(H) = H \cup \{0\}$ so that $b(H) = H \cup \{0\}$.

(iv) $\text{Int}\, \mathbf{Z} = \varnothing$ and $Cl(\mathbf{Z}) = b(\mathbf{Z}) = \mathbf{Z}$. ■

SOLUTION TO SAQ 15

(i) If $H = (-\infty, 0]$ then $\text{Int}\, H = (-\infty, 0)$ and $Cl(H) = \mathbf{R}$ so that $b(H) = [0, \infty)$.

(ii) If $H = [0, 1]$ then $\text{Int}\, H = \varnothing$ and $Cl(H) = b(H) = [0, \infty)$.

(iii) If $H = [1, \infty)$ then $\text{Int}\, H = \varnothing$ and $Cl(H) = b(H) = [1, \infty)$.

(iv) $\text{Int}\, \mathbf{Z} = \varnothing$ and $Cl(\mathbf{Z}) = b(\mathbf{Z}) = \mathbf{R}$. ■

SOLUTION TO SAQ 16 By Proposition 4, $Cl(A - H) = A - \text{Int}\, H$, so that

$$\text{Int}\, H = A - Cl(A - H).$$

But, as shown in Example 7,

$$Cl(A - H) = \{x \in A : d(x, A - H) = 0\}.$$

Since $d(x, A - H) \geq 0$ for all x in A it follows that

$$\text{Int}\, H = \{x \in A : d(x, A - H) \neq 0\} = \{x \in A : d(x, A - H) > 0\}.$$

Also, $b(H) = \{x \in A : x \in Cl(A) \text{ and } x \in Cl(A - H)\}$

$$= \{x \in A : d(x, A) = 0 \text{ and } d(x, A - H) = 0\}.$$

(In other words, the boundary of a set consists of those points at a zero distance from both the set and its complement.) ■

5.4 Dense and Nowhere-dense Sets

The rational density property has played an important role in many of our discussions. A restatement of this property is that in $\mathbf{R}$ with the usual topology we have $Cl(\mathbf{Q}) = \mathbf{R}$. The property of a set being everywhere-dense is a generalization of this idea to arbitrary topological spaces. In the case of those spaces which are metrizable, an everywhere-dense set nearly fills out all of the space. At the other extreme, we shall study nowhere-dense sets which, in a metrizable space, are very sparse sets.

READ S Page 63, Definition 3.7.20 and Example 3.7.21, and page 64, lines 16 to -5.

NOTES (i) *Page 63, Definition 3.7.20*

It is a consequence of S Proposition 3.7.12 that a subset H of a topological space T is dense in T if and only if $V \cap H \neq \varnothing$ for every non-empty open set V of T.

For if H is dense in T, V is a non-empty open set of T and $x \in V$, then $x \in Cl(H) = T$, so that $V \cap H \neq \varnothing$ by S Proposition 3.7.12.

Conversely, let H be a subset of T with the property that $V \cap H \neq \varnothing$ for every non-empty open set V of T. If x is a point of T and V is an open set containing x, then $V \cap H \neq \varnothing$ so that $x \in Cl(H)$ by **S** Proposition 3.7.12. Thus $Cl(H) = T$ so that H is dense in T.

(ii) *Page 63, Example 3.7.21*
This example shows that a subset and its complement can both be dense in a topological space. For each point of $\mathbf{R}$ is a limit point of $\mathbf{R} - \mathbf{Q}$ by the irrational density property (*Unit 2* Section 2.1) and hence $Cl(\mathbf{R} - \mathbf{Q}) = \mathbf{R}$. Therefore both $\mathbf{Q}$ and $\mathbf{R} - \mathbf{Q}$ are dense in $\mathbf{R}$. In Example 12 below, we shall see that $\mathbf{R}$ contains a countably infinite set of disjoint dense subsets.

(iii) *Page 64, Definition 3.7.26*
A set need not be either dense or nowhere-dense; for example, $[0, 1]$ in $\mathbf{R}$ with the usual topology.

(iv) *Page 64, Corollary 3.7.29*
S points out that the hypothesis that H is closed cannot be deleted. Notice, however, that the result still holds one way round; in other words, *for any nowhere-dense set H in a topological space T, $T - H$ is dense in T.*

For by **S** Proposition 3.7.28, we have $T = Cl(T - Cl(H))$. But $H \subset Cl(H)$ so that $T - H \supset T - Cl(H)$ and hence $Cl(T - H) = T$. ▲

EXAMPLE 11 Let $T = \{A, \mathscr{T}\}$ be the topological space introduced in Example 1. Recall that $A = \{a, b, c\}$ and that the closed sets of T are A, $\{b, c\}$, $\{a\}$ and $\varnothing$. In Example 4, we saw that

$$Cl(\{a, b\}) = A \quad \text{and} \quad Cl(\{a, c\}) = A.$$

Thus the subsets $\{a, b\}$ and $\{a, c\}$ are dense in T.

In Example 9, we saw that the only non-empty subsets of T with empty interior are $\{b\}$ and $\{c\}$. Since neither of these sets is closed in T, and so is not the closure of any set, there is no non-empty subset H of T such that $\mathrm{Int}(Cl(H)) = \varnothing$. Thus T contains no non-empty nowhere-dense subset.

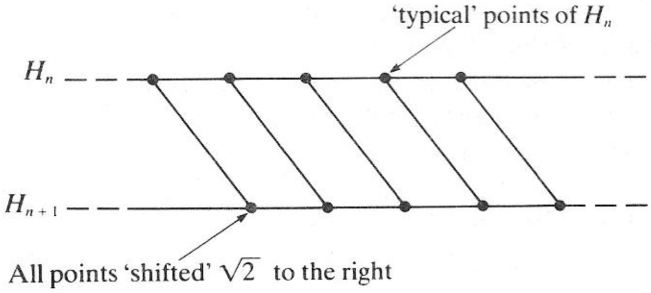
EXAMPLE 12 For each positive integer n let

$$H_n = \{q + n\sqrt{2} : q \in \mathbf{Q}\}.$$

We shall show that H_n is dense in $\mathbf{R}$ (with the usual topology).

We must show that $Cl(H_n) = \mathbf{R}$ or, equivalently, that for each real number a and each positive number ε, the interval $(a - \varepsilon, a + \varepsilon)$ contains a point of H_n. Since $a - \varepsilon - n\sqrt{2} < a + \varepsilon - n\sqrt{2}$, by the rational density property (*Unit 2* Section 2.1) there exists a rational number q such that

$$a - \varepsilon - n\sqrt{2} < q < a + \varepsilon - n\sqrt{2}.$$



'typical' points of H_n

All points 'shifted' $\sqrt{2}$ to the right

Thus $a - \varepsilon < q + n\sqrt{2} < a + \varepsilon$ so that $(a - \varepsilon, a + \varepsilon) \cap H_n \neq \varnothing$, as required. Thus H_n is dense in $\mathbf{R}$ for each n in $\mathbf{N}$.

Notice that if $n, m \in \mathbf{N}$ and $n \neq m$, then H_n and H_m are disjoint. For suppose that $H_n \cap H_m \neq \varnothing$. Then there exist rationals q and r such that $q + n\sqrt{2} = r + m\sqrt{2}$, which gives

$$\sqrt{2} = \frac{r - q}{n - m}.$$

Since $(r - q)/(n - m)$ is a rational number, this is absurd. Hence $H_n \cap H_m = \varnothing$. Thus $(H_n)_{n \in \mathbf{N}}$ is a countably infinite family of disjoint dense subsets of **R**!

EXAMPLE 13 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the left-ray topology on **R** studied in Example 2. In SAQ 8, you showed that if H is a subset of **R** which is not bounded below, then $Cl(H) = \mathbf{R}$; whilst if H is bounded below, then $Cl(H) = [b, \infty)$ where $b = \inf H$. Therefore each subset of **R** which is not bounded below is dense in T. For example, **Z** is dense in T and $\{n \in \mathbf{Z} : n \leq -10^6\}$ is dense in T.

On the other hand, if H is a subset which is bounded below and $Cl(H) = [b, \infty)$, then $T - Cl(H) = (-\infty, b)$, which is not bounded below, so that $T - Cl(H)$ is dense in T. It follows from **S** Proposition 3.7.28 that if H is not bounded below then H is nowhere-dense in T. We notice that for this topological space T, every subset is either dense in T or nowhere-dense in T.

SAQ 17 Prove that if H_1 and H_2 are dense subsets of topological spaces T_1 and T_2 respectively, then $H_1 \times H_2$ is dense in the topological product $T_1 \times T_2$. Exhibit a countably infinite dense subset of $\mathbf{R}^2$ (with the usual topology).

SAQ 18 Prove that if H and K are dense subsets of a topological space T, and H is open in T, then $H \cap K$ is dense in T. [Hint: Use Note (i).]

Find dense subsets H and K of **R** (with the usual topology) for which $H \cap K$ is not dense in T.

SAQ 19 Find all proper dense subsets and all non-empty nowhere-dense subsets of the topological space T studied in Example 3.

SOLUTION TO SAQ 17 We saw in SAQ 11 that

$$Cl(H_1 \times H_2) = Cl(H_1) \times Cl(H_2).$$

But since H_i is dense in T_i for $i = 1, 2$, we have $Cl(H_i) = T_i$. Hence

$$Cl(H_1 \times H_2) = T_1 \times T_2,$$

and so $H_1 \times H_2$ is a dense subset of $T_1 \times T_2$.

Here is one example. Since **Q** is a dense subset of **R**, it follows from the first part of the question that $\mathbf{Q} \times \mathbf{Q}$ is dense in $\mathbf{R}^2$. Also (see *Unit 1* Section 1.5), $\mathbf{Q} \times \mathbf{Q}$ is a countably infinite set. ■

SOLUTION TO SAQ 18 Let V be a non-empty open set of T. Then (by Note (i)) $V \cap H \neq \varnothing$. But H is an open set of T, so that $V \cap H$ is a non-empty open set. Therefore $(V \cap H) \cap K \neq \varnothing$. But $(V \cap H) \cap K = V \cap (H \cap K)$, so that $V \cap (H \cap K) \neq \varnothing$ for each non-empty open set V of T. Hence $H \cap K$ is dense in T.

Let $H = \mathbf{Q}$, $K = \mathbf{R} - \mathbf{Q}$; then H, K are dense in **R**, but $H \cap K = \varnothing$ so $H \cap K$ is not dense in **R**. ■

SOLUTION TO SAQ 19 We showed in Example 6 that if H is not a closed set of T then $Cl(H) = H \cup \{0\}$. It follows that $\mathbf{R} - \{0\}$ is the only proper dense subset of T. It now follows from **S** Proposition 3.7.28 that a non-empty subset H of T is nowhere-dense in T if and only if $Cl(H) = \{0\}$ and hence $H = \{0\}$. ■

5.5 Hausdorff Spaces

In this section, we study topological spaces which satisfy a condition, called the *Hausdorff condition*, which (as we have already seen in SAQ 19 of *Unit 3*) holds in every metric space.

READ S Page 72, line 1 to page 73, line −7.

NOTES (i) *Page 72, Definition 4.1.1*

We recall (**S** Definition 1.2.2) that in **R** a sequence (x_n) converges to x if given any $\varepsilon > 0$ there exists a positive integer N such that $|x - x_n| < \varepsilon$ for all $n \geq N$. The natural generalization of this idea to metric spaces is to say that a sequence (x_n) of points of a metric space M, with metric d, converges to a point x of M if given any real number $\varepsilon > 0$, there exists a positive integer N such $d(x, x_n) < \varepsilon$ for all $n \geq N$. Since

$$B_\varepsilon(x) = \{y \in M : d(x,y) < \varepsilon\},$$

this is precisely the content of Definition 4.1.1.

(ii) *Page 73, Definition 4.2.1*

It is a consequence of the definition that if T is a Hausdorff space and x is a point of T, then $\{x\}$ is a closed set of T. For if $y \in T - \{x\}$, then x and y are distinct points of T so that there exist disjoint open sets U, V of T such that $x \in U$ and $y \in V$. Since $U \cap V = \varnothing$ and $x \in U$, it follows that $x \notin V$ and hence $V \subset T - \{x\}$. Therefore $T - \{x\}$ is an open set of T and so $\{x\}$ is a closed set of T, as asserted.

It follows by **S** Proposition 3.7.4 that every finite subset of a Hausdorff space T is a closed set of T.

(iii) *Page 73, Example 4.2.3*

You showed that every metric space is a Hausdorff space in SAQ 19 of *Unit 3*.

To show, in Note (iv) of *Unit 4* Section 4.1, that the topological space $T = \{\mathbf{R}, \mathscr{T}\}$, introduced in **S** Example 3.1.7, is not a metrizable space, we showed, in fact, that it is not a Hausdorff space. Recall that $\mathscr{T}$ consists of all subsets U of **R** such that $\mathbf{R} - U$ is finite. Thus the proper closed subsets of T are the finite subsets. In particular, each singleton subset $\{x\}$, $x \in \mathbf{R}$, of T is closed. Thus T is an example of a topological space in which every singleton subset is closed, but which is not a Hausdorff space.

(iv) *Page 73, Proposition 4.2.4*

The proof of part (a) of this proposition goes as follows. Let T be a Hausdorff space, H a subspace of T and x, y distinct points of H. Then x, y are distinct points of T, so there exist disjoint open sets U, V of T containing x, y respectively. Then $U \cap H$ and $V \cap H$ are disjoint open sets of H containing x and y respectively. Thus H is a Hausdorff space.

In SAQ 32 you are asked to prove part (b) of this proposition, and in SAQ 21 to prove part (c). The meaning and significance of part (d) of the proposition will be explained in Section 5.6. ▲

EXAMPLE 14 Consider the topological space $T = \{A, \mathscr{T}\}$ where $A = \{a,b,c\}$ and

$$\mathscr{T} = \{\varnothing, \{a\}, \{b,c\}, A\}.$$

We notice that the open sets of T which contain b, namely $\{b,c\}$ and A, also contain c. Thus T is not a Hausdorff space since we cannot find disjoint open sets U, V containing b, c respectively.

Remark. If $T = \{A, \mathscr{T}\}$ where A is a finite set and T is a Hausdorff space, then T is a discrete space. For, as pointed out in Note (ii), every finite subset of a Hausdorff space is closed. Thus every subset of the finite space T is closed and hence every subset of T is open, i.e. T is a discrete space.

EXAMPLE 15 Let $T = \{\mathbf{R}, \mathcal{T}\}$ where $\mathcal{T}$ consists of all subsets V of $\mathbf{R}$ such that either $0 \notin V$ or $\mathbf{R} - V$ is finite. Then T is a Hausdorff space.

To see this, let x and y be distinct real numbers. If $x \neq 0$ and $y \neq 0$ then taking $U = \{x\}$ and $V = \{y\}$ we obtain disjoint open sets U, V containing x, y respectively. If $y = 0$ then $x \neq 0$, and by taking $U = \{x\}$ and $V = \mathbf{R} - \{x\}$ we obtain disjoint open sets containing x, y respectively. Hence combining the two cases, we can conclude that T satisfies the Hausdorff condition.

SAQ 20 Let $T = \{A, \mathcal{T}\}$ where $A = \{a, b, c\}$ and $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, A\}$, and consider the sequence (x_n) in T where

$$x_n = \begin{cases} a, & \text{if } n \text{ is odd,} \\ b, & \text{if } n \text{ is even.} \end{cases}$$

Prove that (x_n) converges both to b and to c, but does not converge to a.

Remark. It follows from Proposition 4.2.2 that T is not a Hausdorff space.

SAQ 21 Let T_1 be a topological space, T_2 a Hausdorff space and $f: T_1 \longrightarrow T_2$ an injective and continuous function. Prove that T_1 is a Hausdorff space.

***SAQ 22** Let S be a topological space, T a Hausdorff space and $f, g: S \longrightarrow T$ continuous functions. Prove that

$$\{x \in S: f(x) = g(x)\}$$

is a closed set of S. Deduce that, if there exists a dense subset D of S such that $f|D = g|D$, then $f = g$.

Remark. This last result is a generalization of the result in SAQ 17 of *Unit 2*, where $S = T = \mathbf{R}$ with the usual topology.

SOLUTION TO SAQ 20 The open sets containing b are $\{a, b\}$ and A. Thus if U is an open set containing b then $x_n \in U$ for every n. Hence the condition of S Definition 4.1.2 is satisfied and so (x_n) converges to b. Similarly, since the only open set containing c is A, the sequence (x_n) converges to c.

The sequence (x_n) does not converge to a since $U = \{a\}$ is an open set of T which contains a but (since $x_n = b$ for every even positive integer n) there is no positive integer N such that $x_n \in U$ for all $n \geq N$. ■

SOLUTION TO SAQ 21 Let x and y be distinct points of T_1. Then $f(x)$ and $f(y)$ are distinct points of T_2, since f is an injective function. Since T_2 is a Hausdorff space, there exist disjoint open sets U and V of T_2 such that $f(x) \in U$ and $f(y) \in V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets of T_1, since f is a continuous function, and $x \in f^{-1}(U)$, $y \in f^{-1}(V)$. Also

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\varnothing) = \varnothing.$$

Therefore $f^{-1}(U)$, $f^{-1}(V)$ are disjoint open sets of T_1 containing x, y respectively. ■

SOLUTION TO SAQ 22 Let

$$E = \{x \in S: f(x) = g(x)\},$$

and let a be a point of $S - E$. Then $f(a)$ and $g(a)$ are distinct points of T. Since T is a Hausdorff space, there exist disjoint open sets U and V of T such that $f(a) \in U$ and $g(a) \in V$. Since f and g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open sets of S containing a. Thus, if we put

$$W = f^{-1}(U) \cap g^{-1}(V),$$

then W is an open set of S and $a \in W$. Now if $x \in W$ then $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$ so that $f(x) \in U$ and $g(x) \in V$; therefore $f(x) \neq g(x)$, since U and V are disjoint, and hence $x \in S - E$. Hence, for each point a of $S - E$, there exists an open set W of S such that $a \in W \subset S - E$. Therefore $S - E$ is open, and so E is a closed set of S.

Now if there exists a set D that is dense in S such that $f|D = g|D$, then $D \subset E$ so that $Cl(D) \subset E$, since E is a closed set. But $Cl(D) = S$ since D is dense in S, so that $E = S$. Therefore $f(x) = g(x)$ for all $x \in S$, i.e. $f = g$. ■

5.6 Homeomorphisms

READ S all of Section 3.6, from page 56.

NOTES (i) *Page 56, Definition 3.6.1*
We have already met the idea of a homeomorphism or topological equivalence between metric spaces in *Unit 3* Section 3.6 (**S** Definition 2.4.7). A homeomorphism between metric spaces is simply a homeomorphism between the associated topological spaces.

(ii) *Page 57, lines 1 to 5*
In SAQ 10 of *Unit 3*, you found an example of a continuous bijection which is not a homeomorphism. This leads us to ask what additional condition a continuous bijection f must satisfy to ensure that f^{-1} is continuous, i.e. to ensure that f is a homeomorphism.

PROPOSITION **8** Consider a bijection $f: T_1 \longrightarrow T_2$ where T_1 and T_2 are topological spaces. The following statements are equivalent:

(a) f is a homeomorphism; and
(b) a subset U of T_1 is open in T_1 if and only if $f(U)$ is open in T_2.

In other words, f is a homeomorphism if and only if f is a one-one correspondence between points which is at the same time a one-one correspondence between the open sets.

PROOF To see that (a) implies (b), let $f: T_1 \longrightarrow T_2$ be a homeomorphism and consider a subset U of T_1. If $f(U)$ is open in T_2 then U is open in T_1, for $U = f^{-1}(f(U))$ and f is continuous. Also, if U is open in T_1 then $f(U)$ is open in T_2, for $f(U) = (f^{-1})^{-1}(U)$ and f^{-1} is continuous. Thus (b) holds.

Conversely, let $f: T_1 \longrightarrow T_2$ be a bijection which satisfies condition (b). If V is an open set of T_2 then, since $f(f^{-1}(V)) = V$, it follows that $f^{-1}(V)$ is open in T_1. Thus f is continuous. Also, if U is open in T_1, then, since $(f^{-1})^{-1}(U) = f(U)$, it follows that $(f^{-1})^{-1}(U)$ is open in T_2. Thus $f^{-1}: T_2 \longrightarrow T_1$ is continuous. Therefore f is a bijection such that both f and f^{-1} are continuous and so condition (a) holds. ■

NOTES (iii) *Page 57, line 7*
Let us show that topological equivalence is an equivalence relation on the collection of topological spaces.

The relation is reflexive, for the identity map is a homeomorphism of a topological space with itself.

Next, if $f: T_1 \longrightarrow T_2$ is a homeomorphism between topological spaces T_1 and T_2 then

$$f^{-1}: T_2 \longrightarrow T_1$$

is a bijection which is continuous. Also, $(f^{-1})^{-1} = f$ so that $(f^{-1})^{-1}$ is continuous and hence f^{-1} is a homeomorphism between T_2 and T_1. Thus the relation is symmetric.

Finally, let $f: T_1 \longrightarrow T_2$ and $g: T_2 \longrightarrow T_3$ be homeomorphisms between topological spaces T_1, T_2, T_3, and put $h = g \circ f$. Then h is a bijection and $h^{-1} = f^{-1} \circ g^{-1}$. It follows from **S** Proposition 3.1.8 that both h and h^{-1} are continuous. Hence

$$h: T_1 \longrightarrow T_3$$

is a homeomorphism. Thus the relation of topological equivalence is transitive, and hence is an equivalence relation.

(iv) *Page 57, Example 3.6.2(a)*
The inverse function of f is

$$x \longmapsto a + ((b - a)(x - c)/(d - c)), \quad x \in (c, d).$$

In fact, in SAQ 22 of *Unit 3*, you showed that any open interval in $\mathbf{R}$ is homeomorphic to $(0, 1)$. Since homeomorphism is an equivalence relation, it follows that any two open intervals in $\mathbf{R}$ are homeomorphic.

(v) *Page 58, Proposition 3.6.3*
The existence of the bijection ψ was established in Proposition 10 of *Unit 1*; see also the Remark following that proposition.

(vi) *Page 59, lines 6 to 20*
A homeomorphism $f: S \longrightarrow T$ determines a one-one correspondence between the points of S and T and also, as seen in Note (ii), between the topologies of S and T. Therefore every property defined in terms of open sets and in terms of set theory is a topological property.

We note that the property of being a Hausdorff space is a topological property. For let $f: T_1 \longrightarrow T_2$ be a homeomorphism, and suppose that T_2 is a Hausdorff space. Then f is injective and continuous, so that T_1 is a Hausdorff space by **S** Proposition 4.2.4(c). ▲

EXAMPLE 16 In SAQ 2(ii) of *Unit 4*, we considered all possible topologies on the set $A = \{a, b, c\}$. Recall that, in the solution, we said that topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on A are of the same type if and only if there exists a permutation f of A with the property that $V \in \mathscr{T}_1$ if and only if $f(V) \in \mathscr{T}_2$. Thus the topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ are of the same type if and only if there exists a one-one correspondence $f: A \longrightarrow A$ with the property that V is an open set of $\{A, \mathscr{T}_1\}$ if and only if $f(V)$ is an open set of $\{A, \mathscr{T}_2\}$. Now, from Note (ii), we can see that topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on A are of the same type if and only if the topological spaces $\{A, \mathscr{T}_1\}$ and $\{A, \mathscr{T}_2\}$ are homeomorphic. Thus, with respect to the relation of topological equivalence, there are 9 distinct equivalence classes of topological spaces with underlying set A.

EXAMPLE 17 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ consists of all subsets V of $\mathbf{R}$ such that either $0 \notin V$ or $\mathbf{R} - V$ is finite. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be a bijection such that $f(0) = 0$. (For example, f might be given by $f(x) = x^3$, $x \in \mathbf{R}$, or by $f(x) = kx$, $x \in \mathbf{R}$, where $k \neq 0$.) Then $f: T \longrightarrow T$ is a homeomorphism.

For let V be a subset of $\mathbf{R}$. Then since $f(0) = 0$, it follows that $0 \notin V$ if and only if $0 \notin f(V)$. Also, since f is a bijection,

$$\mathbf{R} - f(V) = f(\mathbf{R} - V),$$

and so (as in Note (ii) of Section 5.1) $\mathbf{R} - f(V)$ is finite if and only if $\mathbf{R} - V$ is finite. Thus f is a one-one correspondence with the property that V is an open set of T if and only if $f(V)$ is an open set of T. From Note (ii), we see that $f: T \longrightarrow T$ is a homeomorphism.

EXAMPLE 18 Recall (**S** page 46) that a topological space $T = \{A, \mathscr{T}\}$ is said to be metrizable if there exists a metric d on A such that $\mathscr{T}$ is the set of d-open subsets of A. Let us prove that *metrizability is a topological property*. (In other words, if two topological spaces are homeomorphic and one of them is metrizable, then so is the other.)

Let $f: S \longrightarrow T$ be a homeomorphism where T is a metrizable space, and let d be a metric which gives rise to the topology of T. We must show that S is a metrizable space.

We shall show that:

(i) $\rho: S \times S \longrightarrow \mathbf{R}$ defined by

$$\rho(x, y) = d(f(x), f(y)), \quad (x, y) \in S \times S,$$

is a metric on S;

(ii) the open sets of S are the same as the ρ-open sets of the metric space (S, ρ).

For (i) we first note that $\rho(x,y) = d(f(x), f(y)) \geq 0$ for all x, y in S. Also if $x, y \in S$ and $\rho(x,y) = 0$, then $d(f(x), f(y)) = 0$, so that $f(x) = f(y)$, by condition (M1) for d. But f is a bijection, so that $x = y$. Hence ρ satisfies condition (M1). The verification of (M2) and (M3) for ρ is immediate from the corresponding properties of d.

For (ii), we first show that any ρ-open ball $B_\varepsilon(a;\rho)$ is open in S.

If $x, a \in S$ and $\varepsilon > 0$, then $\rho(x,a) < \varepsilon$ if and only if $d(f(x), f(a)) < \varepsilon$. Therefore, for each $a \in S$ and each $\varepsilon > 0$,

$$f(B_\varepsilon(a;\rho)) = B_\varepsilon(f(a);d).$$

Since $B_\varepsilon(f(a);d)$ is an open set of T and f is a homeomorphism, it follows that $B_\varepsilon(a;\rho)$ is an open set of S. It follows that every ρ-open set (being the union of ρ-open balls) is open in S.

Now let U be an open set of S and let $a \in U$. Then $f(U)$ is an open set of T and $f(a) \in f(U)$. Hence there exists $\varepsilon > 0$ such that

$$B_\varepsilon(f(a);d) \subset f(U).$$

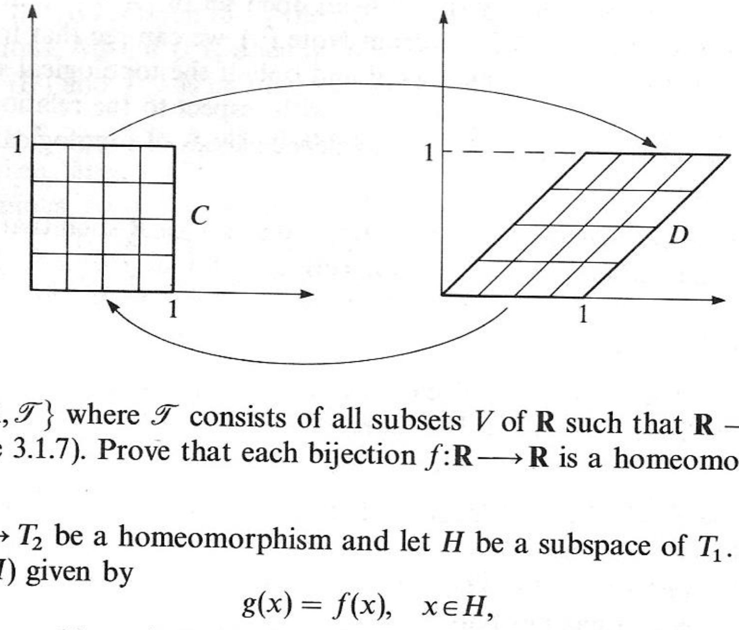
It follows that $B_\varepsilon(a;\rho) \subset U$. Therefore each open set of S is ρ-open.

This completes the proof that the topology of S is the metric topology associated with ρ. Hence S is a metrizable space.

SAQ 23 Show that the subspaces $C = [0,1] \times [0,1]$ and

$$D = \{(x_1, x_2) \in \mathbf{R}^2 : 0 \leq x_2 \leq 1,\ x_2 \leq x_1 \leq x_2 + 1\}$$

of $\mathbf{R}^2$ (with the usual topology) are homeomorphic.



SAQ 24 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ consists of all subsets V of $\mathbf{R}$ such that $\mathbf{R} - V$ is finite (S Example 3.1.7). Prove that each bijection $f: \mathbf{R} \longrightarrow \mathbf{R}$ is a homeomorphism of T to itself.

SAQ 25 Let $f: T_1 \longrightarrow T_2$ be a homeomorphism and let H be a subspace of T_1. Prove that $g: H \longrightarrow f(H)$ given by

$$g(x) = f(x), \quad x \in H,$$

is a homeomorphism of H with the subspace $f(H)$ of T_2. Show also that the subspaces $T_1 - H$ and $T_2 - f(H)$ of T_1 and T_2 respectively are homeomorphic.

SAQ 26 Prove that, if S_1, S_2, T_1 and T_2 are topological spaces such that S_i is homeomorphic to T_i for $i = 1, 2$, then the topological products $S_1 \times S_2$ and $T_1 \times T_2$ are homeomorphic.

SAQ 27 Let $T = \{A, \mathscr{T}\}$ be a topological space and let

$$\Delta = \{(x,y) \in A \times A : x = y\}.$$

Show that the subspace Δ of the topological product $T \times T$ is homeomorphic to T.

SOLUTION TO SAQ 23 From the diagram it looks as though one possible homeomorphism is a 'shear'. So, let us consider the function $f: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by

$$f(x_1, x_2) = (x_1 + x_2, x_2), \quad (x_1, x_2) \in \mathbf{R}^2.$$

The map $p_1 \circ f$ from $\mathbf{R}^2$ to $\mathbf{R}$ (where p_1 is the first projection) given by $(x_1, x_2) \longmapsto x_1 + x_2$, $(x_1, x_2) \in \mathbf{R}^2$, is continuous (by SAQ 25(i) of *Unit 4*) and the map $p_2 \circ f = p_2$ from $\mathbf{R}^2$ to $\mathbf{R}$ (where p_2 is the second projection) given by $(x_1, x_2) \longmapsto x_2$ is continuous. It follows from S Proposition 3.5.4 that f is continuous. Similarly, $g: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by

$$g(x_1, x_2) = (x_1 - x_2, x_2), \quad (x_1, x_2) \in \mathbf{R}^2,$$

is continuous.

If $x = (x_1, x_2) \in C$, then $0 \le x_2 \le 1$ and

$$x_2 \le x_1 + x_2 \le x_2 + 1$$

since $0 \le x_1 \le 1$, so that $f(x) \in D$. Hence $f(C) \subset D$ and the map $h: C \longrightarrow D$ given by $h(x) = f(x)$, $x \in C$, is continuous by Note (v) of *Unit 4* Section 4.4. Also, if $y = (y_1, y_2) \in D$ then $0 \le y_2 \le 1$, and $y_2 \le y_1 \le y_2 + 1$ so that $0 \le y_1 - y_2 \le 1$; hence $g(y) \in C$. Thus $g(D) \subset C$ and the map $k: D \longrightarrow C$ given by $k(y) = g(y)$, $y \in D$, is continuous.

Finally, if $x = (x_1, x_2) \in C$ then

$$\begin{aligned}(k \circ h)(x) &= k(h(x)) = k(x_1 + x_2, x_2) \\ &= ((x_1 + x_2) - x_2, x_2) \\ &= (x_1, x_2) = x;\end{aligned}$$

and if $y = (y_1, y_2) \in D$ then

$$\begin{aligned}(h \circ k)(y) &= h(k(y)) = h(y_1 - y_2, y_2) \\ &= ((y_1 - y_2) + y_2, y_2) \\ &= (y_1, y_2) = y.\end{aligned}$$

Therefore $h: C \longrightarrow D$ is a continuous bijection; and $h^{-1} = k$, which is continuous. Thus h is a homeomorphism between C and D. ■

SOLUTION TO SAQ 24

Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be a bijection. If $V \subset \mathbf{R}$ then, since (as in Note (ii) of Section 5.1)

$$\mathbf{R} - f(V) = f(\mathbf{R} - V),$$

we see that $\mathbf{R} - f(V)$ is finite if and only if $\mathbf{R} - V$ is finite. Thus $f(V)$ is open in T if and only if V is open in T. Hence f is a homeomorphism. ■

SOLUTION TO SAQ 25

It is a consequence of Note (v) of *Unit 4* Section 4.4 that $g: H \longrightarrow f(H)$ given by $g(x) = f(x)$, $x \in H$, is continuous. Also, since $f^{-1}(f(H)) = H$, the map $k: f(H) \longrightarrow H$ given by $k(y) = f^{-1}(y)$, $y \in f(H)$, is continuous. But if $x \in H$, then

$$k(g(x)) = k(f(x)) = f^{-1}(f(x)) = x,$$

and if $y \in f(H)$, then

$$g(k(y)) = g(f^{-1}(y)) = f(f^{-1}(y)) = y;$$

so that g is a bijection and $k = g^{-1}$. Thus g is a continuous bijection with continuous inverse, and so g is a homeomorphism between H and $f(H)$.

Moreover, since $f(T_1 - H) = T_2 - f(H)$, that the subspaces $T_1 - H$ and $T_2 - f(H)$ are homeomorphic is seen to be a consequence of the first part of the question, on replacing H by $T_1 - H$. ■

SOLUTION TO SAQ 26

For $i = 1, 2$, let $f_i: S_i \longrightarrow T_i$ be a homeomorphism. Then

$$f_1 \times f_2: S_1 \times S_2 \longrightarrow T_1 \times T_2$$

is a bijection and, by SAQ 21 of *Unit 4*, $f_1 \times f_2$ is continuous. Likewise,

$$f_1^{-1} \times f_2^{-1}: T_1 \times T_2 \longrightarrow S_1 \times S_2$$

is continuous. But it is easy to check that

$$(f_1 \times f_2)^{-1} = f_1^{-1} \times f_2^{-1}.$$

Therefore $f_1 \times f_2$ is a homeomorphism. ■

SOLUTION TO SAQ 27

This is an immediate consequence of S Proposition 3.6.3 since Δ is the graph of the identity map id_T on T. ■

5.7 Objectives of Unit 5

After working through this unit, you should know:

(1) the definition and properties of closed sets in a topological space;

(2) the definition of a limit point of a subset of a topological space;

(3) the definitions of closure, interior and boundary of a subset of a topological space, and the relations between them;

(4) the definitions of dense and nowhere-dense sets in a topological space, and the relations between them;

(5) the definition of a Hausdorff space, and the fact that certain types of topological spaces are actually Hausdorff spaces;

(6) the definitions of a homeomorphism between topological spaces and of a topological property.

5.8 Revision Questions

SAQ 28 Show that there is a basis for the topology of the subspace $\mathbf{Q}$ of $\mathbf{R}$ (with its usual topology), every member of which is both an open set of $\mathbf{Q}$ and a closed set of $\mathbf{Q}$. [Hint: Try sets of the form $(u, v) \cap \mathbf{Q}$ where $u, v \in \mathbf{R} - \mathbf{Q}$, and use the irrational density property.]

SAQ 29 Prove that a point x of a metric space M is a limit point of a subset H of M if and only if there exists a sequence (x_n) of points of $H - \{x\}$ which converges to x.

SAQ 30 Let $T = \{\mathbf{N}, \mathscr{T}\}$ where $\mathscr{T}$ is the topology for $\mathbf{N}$ which consists of $\varnothing$ and the sets

$$V_n = \{k \in \mathbf{N} : k \geq n\}, \quad n \in \mathbf{N}.$$

(i) Find the closed sets of T.

(ii) Find the closure, interior and boundary in T of each subset of $\mathbf{N}$.

(iii) Which subsets of $\mathbf{N}$ are dense in T, and which subsets are nowhere-dense in T?

(iv) Is T a Hausdorff space?

SAQ 31 Prove that, if T_1 and T_2 are Hausdorff spaces, then the topological product $T_1 \times T_2$ is a Hausdorff space.

SAQ 32 Let $\mathscr{T} \subset 2^{\mathbf{R}}$ consist of $\mathbf{R}$ together with all subsets V of $\mathbf{R}$ such that $0 \notin V$.

(i) Prove that $\mathscr{T}$ is a topology on $\mathbf{R}$.

(ii) Find the closure, interior and boundary of every subset of $T = \{\mathbf{R}, \mathscr{T}\}$.

(iii) Show that every bijection $f \colon \mathbf{R} \longrightarrow \mathbf{R}$ such that $f(0) = 0$ is a homeomorphism of T with itself.

SAQ 33 Show that the following property of a topological space T is a topological property: there is no non-empty proper subset of T which is both open in T and closed in T.

SOLUTION TO SAQ 28 Let V be an open set of $\mathbf{Q}$ and let $a \in V$. Then there exists an open set W of $\mathbf{R}$ such that $V = \mathbf{Q} \cap W$. Since $a \in W$ there exists $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset W$. By the irrational density property (*Unit 2* Section 2.1) there exist irrational numbers u_a and v_a such that

$$a - \varepsilon < u_a < a < v_a < a + \varepsilon.$$

If

$$G_a = \{x \in \mathbf{Q} : u_a < x < v_a\},$$

then G_a is an open set of $\mathbf{Q}$, and $a \in G_a \subset V$. Therefore $V = \bigcup_{a \in V} G_a$. Thus the collection of sets of the form $(u, v) \cap \mathbf{Q}$, where u and v are irrational numbers, is a basis for the topology of $\mathbf{Q}$. But

$$(u, v) \cap \mathbf{Q} = [u, v] \cap \mathbf{Q},$$

so that $(u, v) \cap \mathbf{Q}$ is both an open set of the subspace $\mathbf{Q}$ of $\mathbf{R}$ and a closed set in that space. ■

SOLUTION TO SAQ 29 Suppose first that there exists a sequence (x_n) of points of $H - \{x\}$ which converges to x. Then given $\varepsilon > 0$ there exists a positive integer N such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. Thus $B_\varepsilon(x) \cap (H - \{x\}) \neq \varnothing$ for each $\varepsilon > 0$, and so x is a limit point of H.

Now, in the opposite direction, suppose that x is a limit point of H. Then for each positive integer n, $B_{1/n}(x)$ contains a point, x_n, say, of $H - \{x\}$. The sequence (x_n) of points of $H - \{x\}$ thus obtained converges to x. For let $\varepsilon > 0$ be given. There exists a positive integer N such that $\frac{1}{N} < \varepsilon$. If $n \in \mathbf{N}$ and $n \geq N$ then

$$d(x, x_n) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

so that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. ■

SOLUTION TO SAQ 30

(i) The closed sets of T are the complements of the open sets of T. Thus the closed sets of T are $\mathbf{N}$, $\varnothing$ and each subset $\{1, 2, \ldots, n\}$, $n \in \mathbf{N}$.

(ii) Each non-empty open set of T has a finite complement and each proper closed set of T is finite. Thus if H is a subset of $\mathbf{N}$ such that $\mathbf{N} - H$ is infinite, then $\operatorname{Int} H = \varnothing$; and if K is an infinite subset of $\mathbf{N}$ then $Cl(K) = \mathbf{N}$. Now let H be a subset of $\mathbf{N}$; there are 3 cases to consider.

(a) *H is finite.* Let n be the largest element of H. Then

$$\operatorname{Int} H = \varnothing \quad \text{and} \quad Cl(H) = b(H) = \{1, \ldots, n\}.$$

(b) *$\mathbf{N} - H$ is finite* (so that H is infinite). Let n be the largest element of $\mathbf{N} - H$. Then

$$\operatorname{Int} H = V_{n+1}, \quad Cl(H) = \mathbf{N} \quad \text{and} \quad b(H) = \{1, \ldots, n\}.$$

(c) *H and $\mathbf{N} - H$ are both infinite.* Then

$$\operatorname{Int} H = \varnothing \quad \text{and} \quad Cl(H) = b(H) = \mathbf{N}.$$

(iii) A subset H of $\mathbf{N}$ is dense in T if and only if $Cl(H) = \mathbf{N}$. Thus H is dense in T if and only if H is an infinite subset of $\mathbf{N}$.

A subset H of $\mathbf{N}$ is nowhere-dense in T if and only if $\operatorname{Int}(Cl(H)) = \varnothing$. A closed set of T has empty interior provided that it is finite. Thus the nowhere-dense subsets of T are the subsets with finite closure. Hence H is nowhere-dense in T if and only if H is finite.

(iv) If n and m are distinct positive integers and $n < m$, then every open set of T which contains n also contains m. Hence T is not a Hausdorff space. ■

SOLUTION TO SAQ 31 Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be distinct points of the topological product $T_1 \times T_2$. Then either $x_1 \neq y_1$ or $x_2 \neq y_2$.

If $x_1 \neq y_1$ then x_1 and y_1 are distinct points of the Hausdorff space T_1, so that there exist disjoint open sets U_1 and V_1 of T_1 containing x_1, y_1 respectively. Then $U = U_1 \times T_2$ and $V = V_1 \times T_2$ are disjoint open sets of $T_1 \times T_2$. Also $x \in U$ and $y \in V$.

On the other hand, if $x_1 = y_1$ then $x_2 \neq y_2$, and by a similar argument there exist disjoint open sets U, V of $T_1 \times T_2$ containing x, y respectively. Thus $T_1 \times T_2$ is a Hausdorff space. ■

SOLUTION TO SAQ 32

(i) (T1) is satisfied, since $\mathbf{R} \in \mathscr{T}$ by definition, and $\varnothing \in \mathscr{T}$ since $0 \notin \varnothing$. Clearly, any intersection or any union of subsets of $\mathbf{R}$ not containing 0 does not contain 0, so that (T2) and (T3) are also satisfied. Hence $\mathscr{T}$ is a topology on $\mathbf{R}$.

Remark. Notice that all closed sets other than $\varnothing$ contain 0.

(ii) Let H be a subset of $\mathbf{R}$. If $0 \notin H$ then

$$Cl(H) = H \cup \{0\}, \quad \operatorname{Int} H = H, \quad b(H) = \{0\};$$

whilst if $0 \in H$ then

$$Cl(H) = H, \quad \operatorname{Int} H = H - \{0\}, \quad b(H) = \{0\}.$$

(iii) Let $f:\mathbf{R} \longrightarrow \mathbf{R}$ be a bijection such that $f(0) = 0$. Since $f(0) \notin f(U)$ if and only if $0 \notin U$, we see that if $U \subset \mathbf{R}$ then $U \in \mathscr{T}$ if and only if $f(U) \in \mathscr{T}$. Hence f is a homeomorphism. ■

SOLUTION TO SAQ 33

Let $f: T_1 \longrightarrow T_2$ be a homeomorphism and let T_1 have the property under consideration. If V is a subset of T_2 which is open in T_2 and closed in T_2, then, since f is continuous, $f^{-1}(V)$ is open in T_1 and closed in T_1. From our hypotheses, $f^{-1}(V)$ is not a non-empty proper subset of T_1, so that either $f^{-1}(V) = \varnothing$ or $f^{-1}(V) = T_1$. But since f is a bijection, $f^{-1}(V) = \varnothing$ implies that $V = \varnothing$, and on the other hand $f^{-1}(V) = T_1$ implies that $V = T_2$. Thus T_2 has no non-empty proper subset which is both open in T_2 and closed in T_2. ■

UNIT 6

COMPACTNESS

6.0 Introduction

Compact spaces, which are studied in this unit, based on Chapter 5 of **S**, form one of the most important classes of topological spaces.

The notion of a compact topological space is an abstraction of an important property of bounded closed intervals. The *Heine–Borel Theorem* states that if K is a bounded closed interval in $\mathbf{R}$ and $\mathscr{U}$ is a collection of open sets of $\mathbf{R}$ whose union contains K, then there is a finite subset of $\mathscr{U}$ whose union contains K. The conclusion of this theorem motivates the definition of compactness which is discussed in Section 6.2. With this definition, the Heine–Borel Theorem, as we meet it in Section 6.3, becomes the statement that bounded closed intervals are compact. In Section 6.5 we see that a subspace of Euclidean space is compact if and only if it is closed and bounded.

Other significant properties of bounded closed intervals turn out to be consequences of compactness. For example, in Section 6.4 we see that every continuous real-valued function on a compact space is bounded. The class of all topological spaces T such that every continuous real-valued function on T is bounded, is strictly larger than the class of compact spaces. However, if H is a subspace of Euclidean space then every continuous real-valued function on H is bounded if and only if H is compact.

M203 In M203, the property that every continuous real-valued function on H is bounded was used to define compactness of subspaces H of Euclidean space. Thus those subspaces of Euclidean space which are compact in the sense of M203 are precisely those which are compact in the sense of this unit.

6.1 Motivation

READ S all of Section 5.1, beginning on page 75, line 1.

NOTES (i) *Page 76, Step 3*

Here it is assumed that $f: A \to \mathbf{R}$ is continuous on the subspace A of $\mathbf{R}$. Applying the ε-δ definition of continuity at a in A, with $\varepsilon = 1$, gives us an open ball $B_\delta(a)$ *of the subspace* A *of* $\mathbf{R}$ such that $|f(x) - f(a)| < 1$ for all x in $B_\delta(a)$. We recall that

$$B_\delta(a) = A \cap (a - \delta, a + \delta).$$

(ii) *Page 76, Provisional Definition 5.1.3*

To avoid possible misconceptions, we clarify **S**'s provisional definition of compactness for subsets of $\mathbf{R}$: a subset A of $\mathbf{R}$ is compact if whenever A is the union of a collection of open balls *of the subspace* A *of* $\mathbf{R}$ (that is, of open balls with respect to the induced metric on A), it is a union of finitely many balls in the collection.

(iii) *Page 77, line 8*

The provisional definition of compactness for a metric space takes the form: a metric space M is compact if whenever M is the union of a collection of open balls of M, it is a union of finitely many balls in that collection. ▲

EXAMPLE 1 Let $A = (0,1)$ and let $f: A \to \mathbf{R}$ be given by

$$f(x) = \frac{1}{x}, \quad 0 < x < 1.$$

It was shown in **S** Example 5.1.2 that f is not bounded on A. Let us examine why an argument to prove boundedness of f in the way outlined in the reading passage fails for this function f. Our strategy will be to find a collection of open balls of A whose union is A, on each of which f is bounded. We shall find that no union of a finite number of these balls is equal to A. For each a in A, we shall find $\delta(a) > 0$ such that $f(x) < f(a) + 1$ for all x in the open ball $B_{\delta(a)}(a)$ of A; since also $f(x) > 1$ for all x in A, it will follow that f is bounded on each ball $B_{\delta(a)}(a)$.

If $\frac{1}{2} < a < 1$, let us take $\delta(a) = a - \frac{1}{2}$. Then

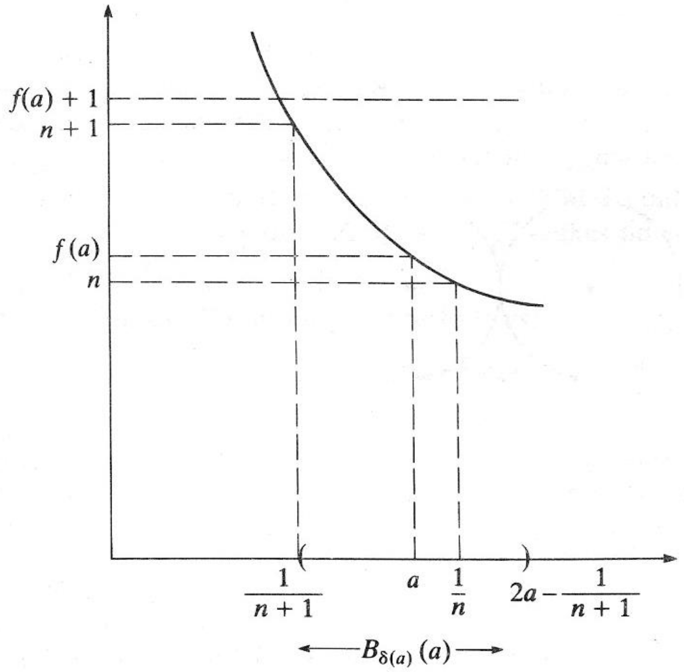
$$B_{\delta(a)}(a) = \begin{cases} (\frac{1}{2}, 2a - \frac{1}{2}), & \text{if } a \leq \frac{3}{4}, \\ (\frac{1}{2}, 1), & \text{if } a > \frac{3}{4}. \end{cases}$$

In both cases, $x > \frac{1}{2}$ if $x \in B_{\delta(a)}(a)$ so that $f(x) < 2 < f(a) + 1$.

For $a < \frac{1}{2}$, if $\dfrac{1}{n+1} < a \leq \dfrac{1}{n}$ where $n \geq 2$, let us take $\delta(a) = a - \dfrac{1}{n+1}$. Then

$$B_{\delta(a)}(a) = \left(\frac{1}{n+1}, 2a - \frac{1}{n+1}\right).$$

If $x \in B_{\delta(a)}(a)$, then $x > \dfrac{1}{n+1}$ so that $f(x) < n + 1$. But $a \leq \dfrac{1}{n}$ so that $f(a) \geq n$ and hence $f(x) < f(a) + 1$ for all x in $B_{\delta(a)}(a)$.



In this way we have defined $\delta(a)$ for each a in A such that $f(x)$ is bounded on each $B_{\delta(a)}(a)$, and

$$A = \bigcup_{a \in A} B_{\delta(a)}(a).$$

Look at *Step 4* of the reading passage. If the union of a *finite* number of the balls were equal to A, then since f is bounded on each of the balls it would follow, as in *Step 2*, that f is bounded on A. But we cannot express A as the union of a finite number of the balls $B_{\delta(a)}(a)$. For suppose that $a_1, \ldots, a_k \in A$ and

$$A = \bigcup_{i=1}^{k} B_{\delta(a_i)}(a_i).$$

Since $a_i > 0$ for $i = 1, \ldots, k$, there exists a positive integer N such that

$$\frac{1}{N+1} < \operatorname{Min}\{a_1, \ldots, a_k\}.$$

Then $B_{\delta(a_i)}(a_i) \subset \left(\dfrac{1}{N+1}, 1\right)$ for $i = 1, \ldots, k$, so that

$$(0, 1) = \bigcup_{i=1}^{k} B_{\delta(a_i)}(a_i) \subset \left(\frac{1}{N+1}, 1\right).$$

This contradiction shows that A is not the union of a finite number of the balls $B_{\delta(a)}(a)$.

6.2 Covers and Compactness

In this section we meet the formal definition of compactness. The definition requires the notion of an open cover of a topological space and of a subcover of a cover.

Now play the first part of the tape for Unit 6

READ S page 77, lines 11 to 21.

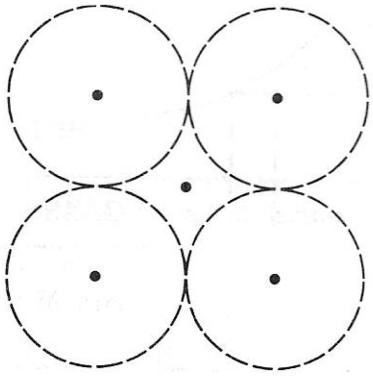
NOTE *Page 77, line 20*

Taking $A = T$ in S Definition 5.2.1, we see that an open cover of a topological space T is a collection $\mathscr{U}$ of sets open in T such that

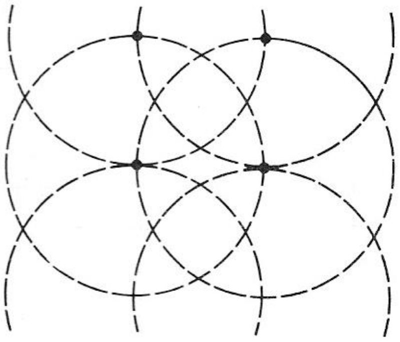
$$T = \bigcup_{U \in \mathscr{U}} U. \quad \blacktriangle$$

EXAMPLE 2 We examine two collections of open sets of $\mathbf{R}^2$, with the usual topology, one of which is a cover of $\mathbf{R}^2$ whilst the other is not.

(i) For each pair (m, n) of integers, let $B_{(m,n)}$ be the open ball in $\mathbf{R}^2$ centre (m, n) and radius $\frac{1}{2}$. Then $(B_{(m,n)})_{(m,n) \in \mathbf{Z} \times \mathbf{Z}}$ is a collection of sets open in $\mathbf{R}^2$ but it is not an open cover of $\mathbf{R}^2$, since, for example, there is no member of the collection which contains the point $(\frac{1}{2}, \frac{1}{2})$.



(i)



(ii)

(ii) Now for each pair (m, n) of integers, let $C_{(m,n)}$ be the open ball in $\mathbf{R}^2$ centre (m, n) and radius 1. Then $(C_{(m,n)})_{(m,n) \in \mathbf{Z} \times \mathbf{Z}}$ is an open cover of $\mathbf{R}^2$, since each point of $\mathbf{R}^2$ is at a distance less than 1 from at least one point both of whose coordinates are integers.

EXAMPLE 3 Here we look at subcovers of an open cover of $\mathbf{R}$, with the usual topology.

Let $\mathscr{U}$ be the collection of all intervals $U_n = (-n, n)$, where $n \in \mathbf{N}$. It is easy to see that $\mathscr{U}$ is an open cover of $\mathbf{R}$. Now let (n_i) be any strictly increasing (infinite) sequence of positive integers. Then the subcollection $\{U_{n_1}, U_{n_2}, \ldots, U_{n_i}, \ldots\}$ of $\mathscr{U}$ is a subcover of $\mathscr{U}$. For if $x \in \mathbf{R}$, there exists a positive integer k such that $k > |x|$. But $n_k \geq k$, since the sequence (n_i) is strictly increasing, so that $|x| < n_k$; from this it follows that $x \in U_{n_k}$. Therefore

$$\mathbf{R} = \bigcup_{k=1}^{\infty} U_{n_k}.$$

We notice that no finite subcollection of $\mathscr{U}$ is a subcover of $\mathscr{U}$. For let $\{U_{n_1}, \ldots, U_{n_k}\}$ be a finite subcollection of $\mathscr{U}$ and suppose, without loss of generality, that $n_1 < n_2 < \cdots < n_k$. Then $U_{n_i} \subset U_{n_k}$ for $i = 1, \ldots, k$ so that

$$\bigcup_{i=1}^{k} U_{n_i} = U_{n_k} \neq \mathbf{R}.$$

Thus no finite subcollection of $\mathscr{U}$ is a cover of $\mathbf{R}$. $\square$

We are now ready for the definition of compactness.

READ S page 77, line -15 to page 79, line -5.

NOTES (i) *Page 77, Definition 5.2.2*

It is sufficient to consider only open covers of T that do *not* include either T or $\varnothing$.

Firstly, we check that

every open cover of T contains a finite subcover (*)

$\Leftrightarrow$ every open cover of T that does not contain T contains a finite subcover. (**)

For, if (*) holds, then (**) also holds since every cover of T that does not contain T is still a cover of T. In the opposite direction, suppose that (**) holds. Then any open cover of T that contains T trivially contains a finite subcover — T itself. So (*) holds. This establishes the equivalence of (*) and (**).

Now let us look at the condition

every open cover of T that does not contain T or $\varnothing$ contains a finite subcover. (***)

It is fairly clear that (**) and (***) are equivalent, since it cannot make any difference to the possibility of an open cover $\mathcal{U}$ of T containing a finite subcover whether or not $\mathcal{U}$ contains $\varnothing$ — for $\varnothing$ makes no contribution to covering any part of T.

Hence we can rephrase Definition 5.2.2 as follows:

Definition A topological space T is compact if and only if every open cover of T that does not contain T or $\varnothing$ contains a finite subcover.

In practice, we shall usually (without mentioning it specifically) use the definition of compactness in this modified form, in order to avoid the trivial cases of covers that contain T, $\varnothing$.

(ii) *Page 78, Remark 5.2.3 (a)*

The point of this remark is that a subspace T of a topological space S is a compact topological space (i.e. T, with the induced topology, is a compact space) if and only if the condition of Definition 5.2.4 is satisfied.

It is a convenient 'abuse of language' to say that T is a compact subset of a topological space S if T, with the induced topology as a subspace of S, is a compact space. ▲

EXAMPLE 4 $\mathbf{R}$, with the usual topology, is not a compact space. Working directly from Definition 5.2.1, we see that one strategy is to find an open cover of $\mathbf{R}$ which has no finite subcover.

We saw in Example 3 that the open cover $\mathcal{U}$ consisting of the intervals $(-n, n)$, $n \in \mathbf{N}$, has this property.

Remark. In SAQ 23 of *Unit 3* we showed that $\mathbf{R}$ is homeomorphic with its subspace $(-1, 1)$. But any two open intervals in $\mathbf{R}$ are homeomorphic (S Example 3.6.2 (a)), so that $\mathbf{R}$ is homeomorphic with $(0, 1)$. We shall see shortly that compactness is a topological property. Hence, since **S** proves that $(0, 1)$ is not compact (S page 77, line -6), this gives another proof that $\mathbf{R}$ is not compact.

EXAMPLE 5 Let us determine when a discrete space T is compact. Since T is a discrete space, each singleton set $\{x\}$, $x \in T$, is open in T. If T is infinite, then the open cover of T by its singleton subsets $\{x\}$, $x \in T$, has no finite subcover.

Thus an infinite discrete space is not compact. We know that any finite topological space is compact (**S**, page 78, line 5). Therefore a discrete space is compact if and only if it is finite.

EXAMPLE 6 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the topology on $\mathbf{R}$ first introduced in Example 3 on page 96 of *Unit 4*. (Recall that a subset V of $\mathbf{R}$ is open in T if and only if either $0 \notin V$ or $\mathbf{R} - V$ is finite.) We shall show that T is a compact space.

Let $\mathscr{U}$ be an open cover of T. We must find a finite subcover of $\mathscr{U}$.

As explained in Note (i), we can assume that $\mathbf{R}$ is not a member of $\mathscr{U}$. First observe that there exists at least one U_0 in $\mathscr{U}$ such that $0 \in U_0$. Since $0 \in U_0$ and U_0 is open in T, it follows that $\mathbf{R} - U_0$ is finite, say $\mathbf{R} - U_0 = \{x_1, \ldots, x_k\}$. Since $\mathscr{U}$ is a cover of T, for each $i = 1, \ldots, k$ there exists U_i in $\mathscr{U}$ such that $x_i \in U_i$. Then $\{U_0, U_1, \ldots, U_k\}$ is a cover of T and so $\mathscr{U}$ has a finite subcover. Therefore T is a compact space. □

Proposition 1 (below) will give a useful characterization of compactness. Since the collection of all open balls is a basis for the topology of a metric space, it shows, in particular, that the provisional definition of compactness of a metric space, formulated in Note (iii) of Section 6.1, is equivalent to **S** Definition 5.2.2 for metric spaces.

PROPOSITION 1 A topological space T is compact if and only if every cover of T by members of a basis for the topology of T has a finite subcover.

PROOF Let $T = \{A, \mathscr{T}\}$ and let $\mathscr{B}$ be a basis for the topology $\mathscr{T}$. Since each member of $\mathscr{B}$ is open in T, if T is compact then every cover by members of $\mathscr{B}$ has a finite subcover.

Now suppose that T has the property that each cover by members of $\mathscr{B}$ has a finite subcover. We must prove that T is compact; that is, that an arbitrary open cover $\mathscr{U}$ of T has a finite subcover.

For each point x of T, choose a U_x in $\mathscr{U}$ such that $x \in U_x$. Since U_x is an open set and $\mathscr{B}$ is a basis for $\mathscr{T}$, there exists B_x in $\mathscr{B}$ such that $x \in B_x \subset U_x$. Then $(B_x)_{x \in T}$ is a cover of T by members of $\mathscr{B}$. By hypothesis, this cover has a finite subcover $\{B_{x_1}, \ldots, B_{x_n}\}$. Since $\bigcup_{i=1}^{n} U_{x_i} \supset \bigcup_{i=1}^{n} B_{x_i} = T$, we see that $\{U_{x_1}, \ldots, U_{x_n}\}$ is a finite subcover of $\mathscr{U}$. Hence T is compact. ■

SAQ 1 Show that $\mathbf{R}^2$, with the usual topology, is not compact.

SAQ 2 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the topology for $\mathbf{R}$ introduced in **S** Example 3.1.7. Show that T is a compact space.

SAQ 3 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the left-ray topology for $\mathbf{R}$ (introduced in SAQ 22 on page 112 of *Unit 4*).

(i) Show that $(0, 1]$ is a compact subset of T.

(ii) Show that $(0, 1)$ is not a compact subset of T.

*(iii) Show that a subset K of T is compact if and only if it is bounded above and contains $\sup K$.

SAQ 4 Let $\mathscr{T}_1$, $\mathscr{T}_2$ be topologies for a set A such that $\mathscr{T}_2 \subset \mathscr{T}_1$.

(i) Prove that if $\{A, \mathscr{T}_1\}$ is a compact space, then $\{A, \mathscr{T}_2\}$ is a compact space.

(ii) Is it true, in general, that, if $\{A, \mathscr{T}_2\}$ is a compact space, then $\{A, \mathscr{T}_1\}$ is a compact space?

SAQ 5 Prove that, if H and K are compact subsets of a topological space T, then $H \cup K$ is a compact subset of T.

SOLUTION TO SAQ 1 To show that $\mathbf{R}^2$ is not compact, we must exhibit an open cover of $\mathbf{R}^2$ which has no finite subcover.

The open cover $\mathscr{C} = (C_{(m,n)})_{(m,n)\in \mathbf{Z}\times\mathbf{Z}}$, introduced in Example 2(ii), has no finite subcover, for no member of $\mathscr{C}$ other than $C_{(m,n)}$ contains the point (m, n) of $\mathbf{R}^2$. Thus if any member of $\mathscr{C}$ is omitted, the resulting collection of sets fails to be a cover of $\mathbf{R}^2$. ■

SOLUTION TO SAQ 2 The method used here is very similar to that employed in Example 6. Let $\mathscr{U}$ be any open cover of T. We must find a finite subcover of $\mathscr{U}$. (We shall assume, following Note (i), that each member of $\mathscr{U}$ is a non-empty proper open subset of T.)

Let U_0 be a member of $\mathscr{U}$. Then $\mathbf{R} - U_0$ is a non-empty finite set, say $\mathbf{R} - U_0 = \{x_1, \ldots, x_k\}$. Since $\mathscr{U}$ is a cover, for $i = 1, \ldots, k$ there exists U_i in $\mathscr{U}$ such that $x_i \in U_i$. Then $\{U_0, U_1, \ldots, U_k\}$ is a finite cover of T. Since each open cover of T has a finite subcover, T is a compact space. ■

SOLUTION TO SAQ 3 (i) We shall apply S Definition 5.2.4.

Let $\mathscr{U}$ be a cover of $(0, 1]$ by sets open in T. (Again, we may assume that $\varnothing, \mathbf{R} \notin \mathscr{U}$.) Each member of $\mathscr{U}$ is of the form $(-\infty, a)$ where $a \in \mathbf{R}$. There exists a member $(-\infty, a_1)$ of $\mathscr{U}$ such that $1 \in (-\infty, a_1)$. But then $(0, 1] \subset (-\infty, a_1)$. Hence there is a finite subcover of $\mathscr{U}$ consisting of the one set $(-\infty, a_1)$. Therefore $(0, 1]$ is a compact subset of T.

(ii) In this case we show that the condition of S Definition 5.2.4 is not satisfied.

We note that

$$(0,1) \subset \bigcup_{n=1}^{\infty} \left(-\infty, 1 - \frac{1}{n}\right);$$

so that the collection of intervals $\left(-\infty, 1 - \frac{1}{n}\right)$, $n \in \mathbf{N}$, is a cover $\mathscr{U}$ of $(0, 1)$ by sets open in T.

The cover $\mathscr{U}$ of $(0, 1)$ has no finite subcover. For let $n_1, n_2, \ldots, n_k$ be positive integers such that $n_1 < n_2 < \cdots < n_k$. Then

$\left(-\infty, 1 - \frac{1}{n_i}\right) \subset \left(-\infty, 1 - \frac{1}{n_k}\right)$ for $i = 1, 2, \ldots, k$, so that

$$\bigcup_{i=1}^{k} \left(-\infty, 1 - \frac{1}{n_i}\right) = \left(-\infty, 1 - \frac{1}{n_k}\right).$$

Whatever the value of k, and so of n_k,

$$(0,1) \not\subset \left(-\infty, 1 - \frac{1}{n_k}\right).$$

Thus no finite subcollection of $\mathscr{U}$ can be a cover of $(0, 1)$. Thus $(0, 1)$ is not a compact subset of T.

(iii) Suppose first that K is not bounded above. Then the collection of intervals $(-\infty, n)$, $n \in \mathbf{N}$, is a cover $\mathscr{U}$ for K by open sets of T which has no finite subcover. For let $n_1, \ldots, n_k$ be positive integers such that $n_1 < n_2 < \cdots < n_k$. Then $(-\infty, n_i) \subset (-\infty, n_k)$ for $i = 1, \ldots, k$, so that

$$\bigcup_{i=1}^{k} (-\infty, n_i) = (-\infty, n_k).$$

But K is not bounded above, so there exists x in K such that $x > n_k$. Thus no finite subcollection of $\mathscr{U}$ can be a cover of K.

Next suppose that K is bounded above and let $b = \sup K$. There are two cases to consider: $b \notin K$ and $b \in K$.

(a) $b \notin K$. By considering the cover of K by the intervals $\left(-\infty, b-\frac{1}{n}\right)$, $n \in \mathbf{N}$, we see, by the argument used in part (ii), that K is not compact.

(b) $b \in K$. Let $\mathscr{U}$ be a cover of K by sets open in T. Then some member U of $\mathscr{U}$ contains b. Now either $U = \mathbf{R}$ or $U = (-\infty, a)$ for some a. If $U = (-\infty, a)$ then since $b \in U$ it follows that $b < a$. But if $x \in K$ then $x \leq b < a$. Hence $K \subset U$. Thus $\mathscr{U}$ has a subcover for K which consists of just one member of $\mathscr{U}$. Therefore K is compact. ■

SOLUTION TO SAQ 4 (i) Let $\mathscr{U}$ be a cover of A by members of $\mathscr{T}_2$. We must show that $\mathscr{U}$ has a finite subcover.

Since $\mathscr{T}_2 \subset \mathscr{T}_1$, each member of $\mathscr{T}_2$ is a member of $\mathscr{T}_1$. Hence $\mathscr{U}$ is an open cover of the compact space $\{A, \mathscr{T}_1\}$ and so has a finite subcover.

(ii) No. If A is an infinite set, it is not true in general that if $\{A, \mathscr{T}_2\}$ is compact and $\mathscr{T}_2 \subset \mathscr{T}_1$ then $\{A, \mathscr{T}_1\}$ is compact.

To see this, take $\mathscr{T}_1$ to be the discrete topology on A; then $\mathscr{T}_2 \subset \mathscr{T}_1$ and $\{A, \mathscr{T}_1\}$ is not compact (Example 5). If A is a finite set, the result is true since every finite topological space is compact. ■

SOLUTION TO SAQ 5 Let $\mathscr{U}$ be a cover of $H \cup K$ by open sets of T. In accordance with S Definition 5.2.4, we must show that $\mathscr{U}$ contains a finite subcover of $H \cup K$. Since $H \subset H \cup K$, $\mathscr{U}$ is a cover, by open sets of T, of the compact subset H of T, and so contains a finite subcover $\mathscr{V}_1$ of H. Similarly, $\mathscr{U}$ contains a finite subcover $\mathscr{V}_2$ of K. Thus $\mathscr{U}$ contains the finite subcover $\mathscr{V}_1 \cup \mathscr{V}_2$ of $H \cup K$. Therefore $H \cup K$ is a compact subset of T. ■

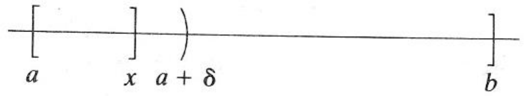
6.3 The Heine–Borel and Bolzano–Weierstrass Theorems

In this section we meet two important theorems about bounded closed intervals in the real line. The Heine–Borel Theorem shows that bounded closed intervals in $\mathbf{R}$ are compact, which, as we shall see later, is the key result for the description of compact sets in the Euclidean spaces $\mathbf{R}^n$ $(n \geq 1)$. The elegant proof of the theorem requires careful study, but mastering it is a worthwhile achievement.

READ S all of Section 5.3, from page 79, line −4 to page 82, line 2.

NOTES (i) *Page 81, line 15*

Let us look a little more closely at the proof that there exists $\delta > 0$ such that $[a, a+\delta) \subset G$.



Note (i)

There exists U in $\mathscr{U}$ such that $a \in U$. Since U is open in $\mathbf{R}$, there exists $\delta > 0$ such that $B_\delta(a) \subset U$, by S Definition 2.3.8. But $B_\delta(a) = (a-\delta, a+\delta)$. Hence $[a, a+\delta) \subset U$. Now if $x \in [a, a+\delta)$ then

$$[a, x] \subset [a, a+\delta) \subset U.$$

Thus if $x \in [a, a+\delta)$ then the closed interval $[a, x]$ is covered by the single member U of $\mathscr{U}$ so that $x \in G$. Therefore $[a, a+\delta) \subset G$.

(ii) *Page 81, lines −14 to −6*

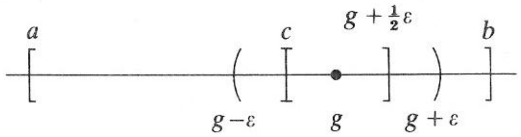
Let us amplify the argument which shows that the assumption that $g \in [a, b]$ leads to a contradiction.

Since $\mathscr{U}$ is a cover of $[a,b]$, if $g \in [a,b]$ then g must belong to some U_0 in $\mathscr{U}$. Since U_0 is open in $\mathbf{R}$, there exists $\varepsilon > 0$ such that $B_\varepsilon(g) \subset U_0$. Now $B_\varepsilon(g) = (g - \varepsilon, g + \varepsilon)$, and there is no loss of generality in supposing that $\varepsilon < g - a$ so that $a < g - \varepsilon < g$. Since $g = \sup G$, there exists a good point c in G such that $g - \varepsilon < c \leq g$. Then

$$g - \varepsilon < c \leq g < g + \tfrac{1}{2}\varepsilon < g + \varepsilon,$$

and we see that

$$[c, g + \tfrac{1}{2}\varepsilon] \subset B_\varepsilon(g) \subset U_0.$$



Note (ii)

But c is good so that $[a,c]$ is covered by a finite subcollection of $\mathscr{U}$, say $\{U_1, \ldots, U_r\}$. Thus

$$[a,c] \subset U_1 \cup \cdots \cup U_r \quad \text{and} \quad [c, g + \tfrac{1}{2}\varepsilon] \subset U_0,$$

so that

$$[a, g + \tfrac{1}{2}\varepsilon] = [a,c] \cup [c, g + \tfrac{1}{2}\varepsilon] \subset U_1 \cup \cdots \cup U_r \cup U_0.$$

Thus the closed interval $[a, g + \frac{1}{2}\varepsilon]$ is covered by the finite subcollection $\{U_1, U_2, \ldots, U_r, U_0\}$ of $\mathscr{U}$. Hence $g + \frac{1}{2}\varepsilon \in G$, contradicting the fact that g is an upper bound for G. ▲

We have just seen that each bounded closed interval in $\mathbf{R}$ is compact. This has the following interesting consequence which is of wide application in many fields of analysis.

ROPOSITION 2 (Bolzano–Weierstrass Theorem) If S is an infinite subset of the bounded closed interval $[a,b]$ in $\mathbf{R}$, then S has a limit point in $[a,b]$.

(The definition of limit point is Definition 3.7.9 on **S** page 61.)

PROOF We prove this by a contradiction argument. Suppose that S has no limit point in $[a,b]$. Then given any point x in $[a,b]$, there exists U_x open in $[a,b]$ such that $x \in U_x$ and U_x contains no points of S except possibly x itself. The family $(U_x)_{x \in [a,b]}$ is an open cover of $[a,b]$ and so, since $[a,b]$ is compact, has a finite subcover, say

$$\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}.$$

Each of the sets U_{x_i} contains at most one point of S and thus their union contains at most n points of S. But since

$$\bigcup_{i=1}^{n} U_{x_i} = [a,b],$$

this is a contradiction, since S is an infinite set. ■

Remarks.

(i) A careful examination of the proof of Proposition 2 shows that the only property of $[a,b]$ that is used is its compactness. So the same proof, with only minor modifications, will yield a more general form of the Bolzano–Weierstrass Theorem: *if S is an infinite subset of a compact topological space T, then S has a limit point in T.* We ask you to prove this result in SAQ 8.

(ii) We can restate this version of the Bolzano–Weierstrass Theorem as:

T compact $\Rightarrow$ every infinite subset of T has a limit point in T.

In SAQ 9, you will find an example of a topological space T with the property that every non-empty subset has a limit point, but T itself is not compact. Thus the converse of the Bolzano–Weierstrass Theorem is false.

SAQ 6 Find a finite subcover of the open cover of $[0,1]$ (with the usual topology) which consists of $[0,\frac{1}{1000})$, $(\frac{1}{2},1]$ and the intervals $\left(\frac{1}{n+2},\frac{1}{n}\right)$, $n\in\mathbf{N}$.

SAQ 7 Find a cover of $[0,1]$ (with the usual topology) by *closed* sets, which has no finite subcover.

SAQ 8 Prove that if S is an infinite subset of a compact topological space T, then S has a limit point in T.

SAQ 9 Let $\mathscr{T}$ be the topology for $\mathbf{N}$ which has as basis the sets

$$\{1,2\},\ \{3,4\},\ \{5,6\},\ldots .$$

Show that:

(i) $T=\{\mathbf{N},\mathscr{T}\}$ is not a compact space; and

(ii) every non-empty subset of T has a limit point.

SOLUTION TO SAQ 6 The only member of the given cover $\mathscr{U}$ which contains 0 is $\left[0,\frac{1}{1000}\right)$, and the only member of $\mathscr{U}$ which contains 1 is $\left(\frac{1}{2},1\right]$. To obtain a finite subcover of $\mathscr{U}$, it remains to find a collection of finitely many members of $\mathscr{U}$ whose union contains $\left[\frac{1}{1000},\frac{1}{2}\right]$. If $n=999$, then $\frac{1}{n+2}<\frac{1}{1000}$ and we see that

$$\left[\frac{1}{1000},\frac{1}{2}\right]\subset\bigcup_{n=1}^{999}\left(\frac{1}{n+2},\frac{1}{n}\right).$$

Thus a finite subcover of $\mathscr{U}$ consists of $\left[0,\frac{1}{1000}\right)$, $\left(\frac{1}{2},1\right]$ and the intervals $\left(\frac{1}{n+2},\frac{1}{n}\right)$ for $n=1,2,\ldots,999$. ■

SOLUTION TO SAQ 7 Each singleton set $\{x\}$ is closed and $[0,1]=\bigcup_{x\in[0,1]}\{x\}$. Plainly this infinite cover has no proper subcover, and therefore no finite subcover. ■.

Remark. Since (as we observed in Note (iii) of Section 5.5 of *Unit 5*) each singleton set in a Hausdorff space is closed, the argument just employed shows that any infinite Hausdorff space has a cover by closed sets which has no finite subcover.

SOLUTION TO SAQ 8 We model our solution on the proof of Proposition 2.

Suppose that S has no limit point in T. Then given any point x in T there exists a set U_x open in T such that $x\in U_x$ and U_x contains no points of S except possibly x itself. The family $(U_x)_{x\in T}$ is an open cover of T and so, since T is compact, has a finite subcover, say

$$\{U_{x_1},U_{x_2},\ldots,U_{x_n}\}.$$

Each of the sets U_{x_i} contains at most one point of S and so their union contains at most n points of S. But since

$$\bigcup_{i=1}^{n}U_{x_i}=T$$

and since S is an infinite set, we have a contradiction. ■

SOLUTION TO SAQ 9 (i) The topological space T is not compact, for

$$\{1,2\},\{3,4\},\{5,6\},\ldots$$

is an open cover of T which has no finite subcover.

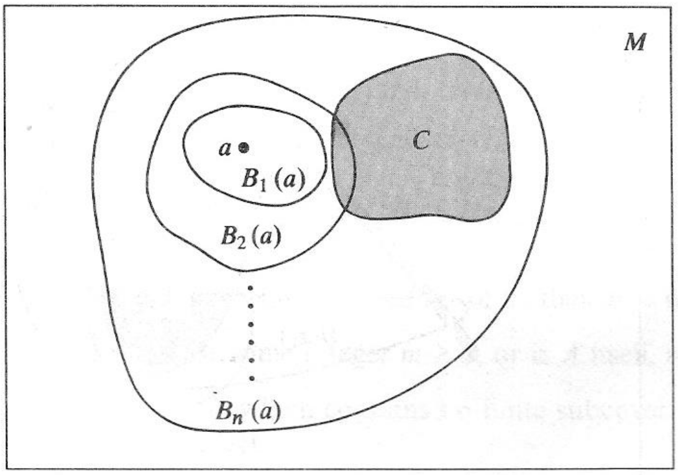
(ii) Let A be a non-empty subset of T and let $k \in A$. If k is odd, then $\{k, k+1\}$ is open in T and every open set of T which contains $k+1$ contains $\{k, k+1\}$. Thus every open set of T which contains $k+1$ contains the element k of A. Hence $k+1$ is a limit point of A. Similarly, if k is even, then $k-1$ is a limit point of A. ■

6.4 Properties of Compact Spaces

READ S Section 5.4, page 82, up to (but including) Definition 5.4.3.

NOTES (i) *Page 82, lines 15 and 16*

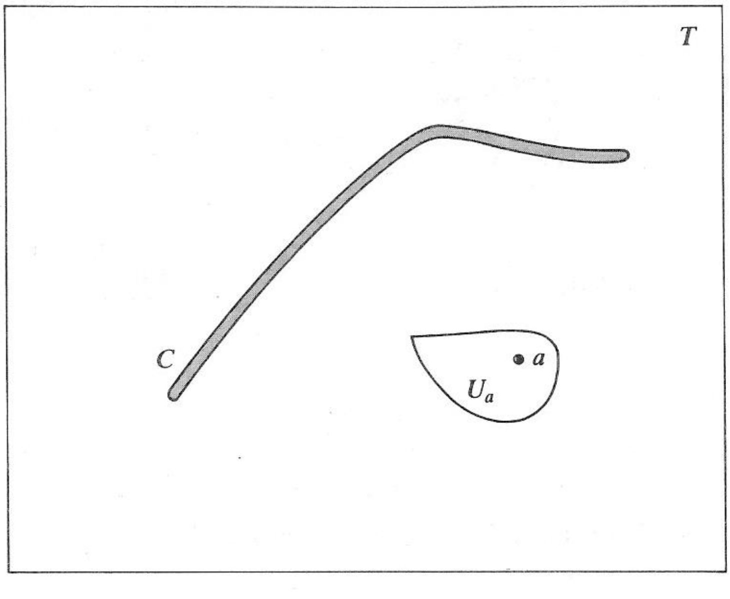
The diagram illustrates the fact that since the sequence $(B_n(a))$ is increasing, the existence of a finite subcover for C ensures that C is bounded.

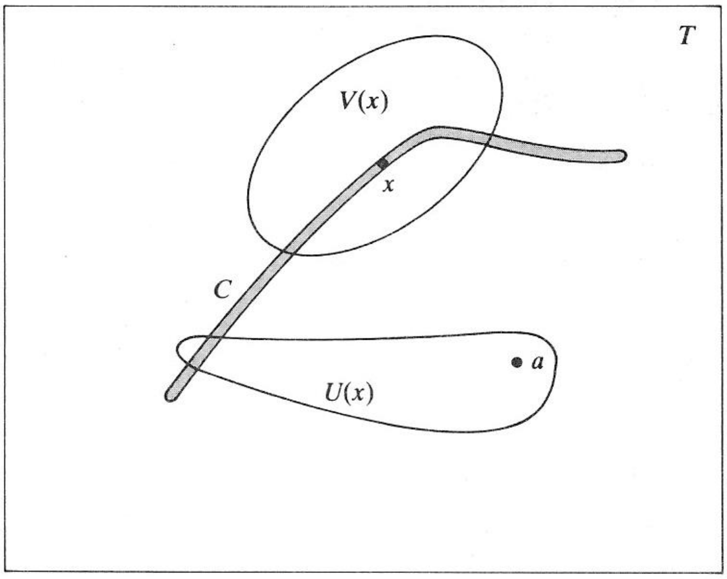


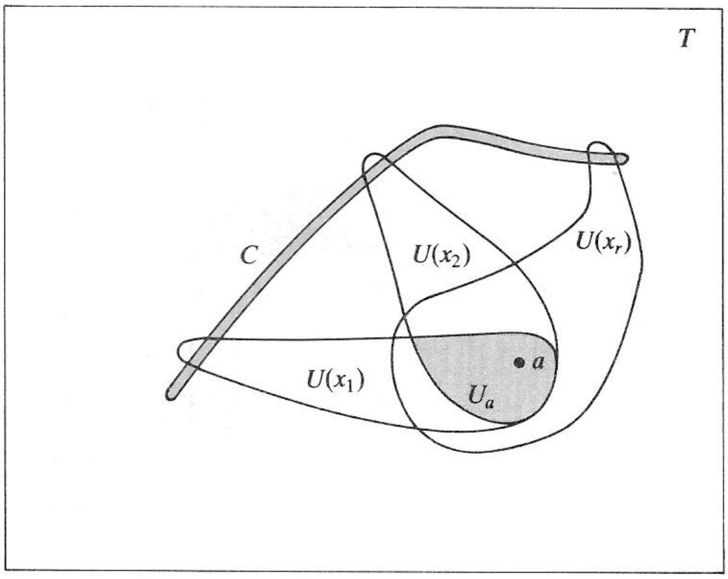
(ii) *Page 82, lines -11, -8, -6; page 83, line 1*

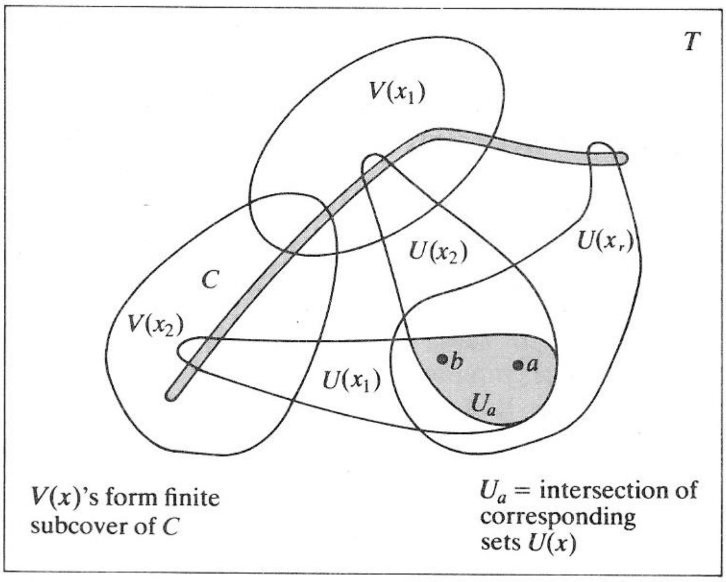
The diagrams below illustrate what the various points and sets in the proof are.

(As we mentioned in the Introduction to the Option, we are assuming that whenever we do not supply such diagrams, you will be sketching them for yourself.)






T
C
U(x_2)
U(x_r)
U(x_1)
a
U_a


T
V(x_1)
U(x_2)
U(x_r)
C
V(x_2)
b
a
U(x_1)
U_a
V(x)'s form finite subcover of C
U_a = intersection of corresponding sets U(x)

(iii) *Page 82, line −11*
The equality

$$T - C = \bigcup_{a \in T-C} U_a$$

follows from the result of SAQ 10 on page 17 of *Unit 1*.

(iv) *Page 82, Propositions 5.4.1 and 5.4.2*
Combining these two results, we see that a compact subset of a metric space is closed and bounded.

The converse of this result is not true. For let M be an infinite discrete metric space (see, for example, **S** Example 2.2.2, page 23). Then every subset of M is closed and bounded; but, as we saw in Example 5, only the finite subsets of M are compact. We shall see later, however, that in Euclidean spaces, the converse is true: that is, every bounded closed subset of $\mathbf{R}^n$ is compact.

(v) *Page 82, Proposition 5.4.2*
This result does not hold in general in spaces which are not Hausdorff. This is shown in a very striking way in Example 7 below. ▲

EXAMPLE 7 Here we describe a topological space in which every proper open set is compact, whilst no non-empty closed set is compact. Let $T = \{A, \mathscr{T}\}$, where A is the open interval $(0, 1)$ in $\mathbf{R}$ and $\mathscr{T}$ consists of $\varnothing$, A and all open intervals $U_n = \left(0, 1 - \frac{1}{n}\right)$ for $n = 2, 3, 4, \ldots$. We leave to you the easy verification that $\mathscr{T}$ is a topology on A.

The non-empty proper open sets of T are the intervals U_n, $n = 2, 3, 4 \ldots$.

First, we show that each U_n is compact. A cover $\mathscr{U}$ for U_n by sets open in T must contain either A or U_m for some $m \geq n$. Thus there is a subcover of the cover $\mathscr{U}$ for U_n which has just one member. By **S** Remark 5.2.3(a), U_n is compact.

Next, we show that if E is a non-empty closed set of T, then E is not compact. Now $E = A - U_m = \left[\frac{1}{m}, 1\right)$ for some integer $m \geq 2$, or is A itself, and $(U_n)_{n \geq 2}$ is a cover for E by sets open in T, which contains no finite subcover. By **S** Remark 5.2.3(a), E is not compact.

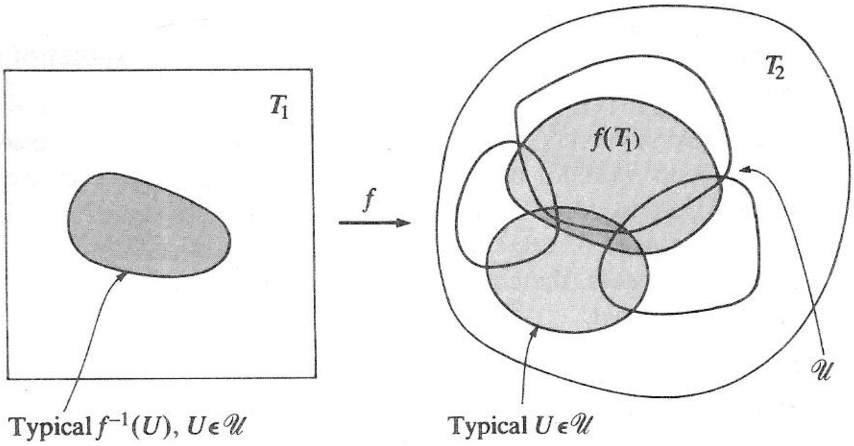
We have shown that T has compact subsets but that no compact subset of T (other than $\varnothing$) is closed. Of course, T is not a Hausdorff space since, for example, every open set of T which contains $\frac{1}{4}$ also contains $\frac{1}{8}$.

SAQ 10 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the topology for $\mathbf{R}$ introduced in **S** Example 3.1.7, page 47. Show that every subspace of T is compact.

SOLUTION TO SAQ 10 Let H be a subspace of T. The open sets of H are of the form $V \cap H$ where V is open in T (by **S** Definition 3.4.1, page 51). Thus the non-empty open sets of H are of the form $H - F$ where F is a finite subset of H. Thus if $\mathscr{U}$ is an open cover of H and $U_0 \in \mathscr{U}$ then $U_0 = H - \{x_1, \ldots, x_k\}$ where $x_i \in H$, $i = 1, \ldots, k$. Now each of these x_i must belong to some set $U_i \in \mathscr{U}$. Thus $\{U_0, \ldots, U_k\}$ is a finite subcover of $\mathscr{U}$. Therefore H is a compact space. ■

READ S from the beginning of Section 5.5 on page 83, to page 84, line −13.

NOTES (i) *Page 83, Proposition 5.5.1*
This theorem is the major result relating continuity and compactness. Notice how simple and elegant the proof is. The following sketch illustrates the relationship between the various sets in the proof.


T_1
f
T_2
$f(T_1)$
$\mathscr{U}$
Typical $f^{-1}(U)$, $U \in \mathscr{U}$
Typical $U \in \mathscr{U}$

(ii) *Page 83, Corollary 5.5.3*

M203

In M203, *Unit 29*, a set $X \subset \mathbf{R}$, $\mathbf{R}^2$ or $\mathbf{R}^3$ is called *compact* if the image of each continuous function $f: X \to \mathbf{R}$ is bounded. You will see in Example 8 below that if we used this definition of compactness for sets in spaces other than $\mathbf{R}^n$ ($n \in \mathbf{N}$) it would not agree with the definition used in **S** (Definition 5.2.4, page 79).

After many years of work by mathematicians, it was generally agreed that S's definition of compactness gave the most useful generalization to *general* topological spaces of the properties of compactness that you have already met in $\mathbf{R}^n$ ($n = 1, 2, 3$).

We now verify that for sets $X \subset \mathbf{R}^n$ ($n = 1, 2, 3$), with the usual topology, the two definitions of compactness coincide.

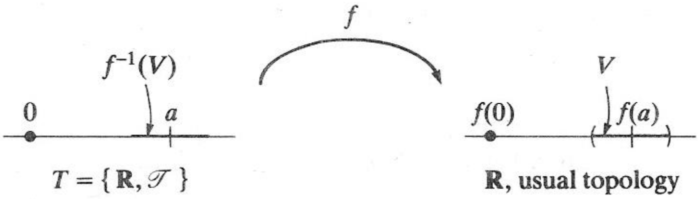
If a set $X \subset \mathbf{R}^n$ ($n = 1, 2, 3$) is compact in the sense of **S**, then every continuous map from X to $\mathbf{R}$ is bounded, by Corollary 5.5.3, since $\mathbf{R}$ is a metric space. Hence X is compact in the M203 sense.

Now let $X \subset \mathbf{R}^n$ be compact in the sense of M203. Then by Theorem 5 of Section 29.2 of M203, X is bounded and closed. It follows from **S** Theorem 5.7.1 (which we shall prove in Section 6.5) that X is compact in the sense of **S**.

(iii) *Page 84, Corollary 5.5.4*
The next example shows that the converse of this result is false. ▲

EXAMPLE 8 There exist topological spaces T which are not compact but which have the property that every continuous real-valued function on T is bounded and attains its bounds. In fact, we shall give an example of a non-compact space on which every continuous real-valued function is constant.

Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ consists of $\varnothing$ and all subsets V of $\mathbf{R}$ such that $0 \in V$. (In other words, $\mathscr{T}$ is the topology which we met, with the different name $\mathscr{T}_0$, in SAQ 6 on page 101 of *Unit 4*.) The topological space T is not compact since the open cover of T by intervals $(-n, n)$, $n \in \mathbf{N}$, has no finite subcover.



Now let f be a real-valued function on T which is not constant; we shall show that f is not continuous. Since f is not constant, there exists a in $\mathbf{R} - \{0\}$ such that $f(a) \neq f(0)$. Now there exists an open interval V of $\mathbf{R}$ such that $f(a) \in V$ and

$f(0) \notin V$. Then $f^{-1}(V)$ is non-empty, since a lies in it; and $f^{-1}(V)$ cannot be an open set in T since $0 \notin f^{-1}(V)$. It follows from the definition of continuous functions that f cannot be continuous. Consequently every continuous real-valued function on T is constant.

SAQ 11 Show that the subspaces $[0,1]$ and $(0,1)$ of $\mathbf{R}$, with the usual topology, are not homeomorphic.

SAQ 12 Show that the topological space T introduced in Example 11 on page 107 of *Unit 4* is not compact. (Hint: You may assume without proof that the subspace H_2 of T is closed and discrete.)

SAQ 13 Let $f: S \to T$ be continuous, where S is a compact space and T is a Hausdorff space.

(a) Show that if E is closed in S, then $f(E)$ is closed in T.

(b) If f is, in addition, an onto function, show that K is closed in T if and only if $f^{-1}(K)$ is closed in S.

SOLUTION TO SAQ 11 We observe that $(0,1)$ is not compact (S page 77, line -6) whilst $[0,1]$ is compact, by the Heine–Borel Theorem. Since compactness is a topological property, we see that $(0,1)$ and $[0,1]$ are not homeomorphic. ■

SOLUTION TO SAQ 12 We argue by contradiction.

Suppose that T is a compact space. Since H_2 is closed in T, it follows from S Proposition 5.6.1 that the subspace H_2 of T is compact. But H_2 is an infinite discrete space and so is not compact, as shown in Example 5.

This contradiction shows that T is not a compact space. ■

SOLUTION TO SAQ 13 (a) Since E is closed in the compact space S, E is compact by S Proposition 5.6.1. Therefore $f(E)$ is compact, by S Proposition 5.5.1. But T is a Hausdorff space, so that $f(E)$ is closed in T, by S Proposition 5.4.2.

(b) If K is closed in T, then $f^{-1}(K)$ is closed in S, by S Proposition 3.7.8, page 61.

Conversely, if K is a subset of T such that $f^{-1}(K)$ is closed in S, then $f(f^{-1}(K))$ is closed in T by part (a). But f is an onto function, so that $K = f(f^{-1}(K))$ by SAQ 14(e) on page 23 of *Unit 1*. ■

In the next reading passage we meet a useful criterion for a continuous bijection to be a homeomorphism. Recall that in SAQ 10 on page 76 of *Unit 3*, we found an example of a continuous bijection which is not a homeomorphism.

READ S all of Section 5.9, beginning on page 88, line 16.

EXAMPLE 9 Let $T = \{A, \mathscr{T}\}$ be a compact Hausdorff space. In this example we shall show that there is no topology $\mathscr{T}_1$ on A, strictly finer than $\mathscr{T}$, such that $\{A, \mathscr{T}_1\}$ is a compact space, and there is no topology $\mathscr{T}_2$ on A, strictly coarser than $\mathscr{T}$, such that $\{A, \mathscr{T}_2\}$ is a Hausdorff space.

So firstly suppose that there exists a topology $\mathscr{T}_1$ on A finer than $\mathscr{T}$ (that is, $\mathscr{T} \subset \mathscr{T}_1$) such that $T_1 = \{A, \mathscr{T}_1\}$ is a compact space. Since $\mathscr{T} \subset \mathscr{T}_1$, the bijection $id_A: T_1 \to T$ is continuous, as pointed out in Note (v) on page 100 of *Unit 4*. Also T_1 is a compact space and T is a Hausdorff space so that, by S Theorem 5.9.1, id_A is a homeomorphism. Therefore $\mathscr{T} = \mathscr{T}_1$. In other words, $\mathscr{T}_1$ cannot be *strictly* finer than $\mathscr{T}$.

Next let $\mathscr{T}_2$ be a topology on A such that $\mathscr{T}_2 \subset \mathscr{T}$ and $T_2 = \{A, \mathscr{T}_2\}$ is a Hausdorff space. Since $\mathscr{T}_2 \subset \mathscr{T}$, the bijection $id_A: T \to T_2$ is continuous; since T is a compact space and T_2 is a Hausdorff space, it follows that id_A is a homeomorphism and hence that $\mathscr{T}_2 = \mathscr{T}$.

Remark. It is possible that there exist non-comparable topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on a set A such that $\{A, \mathscr{T}_1\}$ and $\{A, \mathscr{T}_2\}$ are compact Hausdorff spaces.

For example, let $A = [0,1]$, let $\mathscr{T}_1$ be the usual topology on A and let $\mathscr{T}_2$ be the topology on A as a subspace of the compact Hausdorff space $\{\mathbf{R}, \mathscr{T}\}$ considered in Example 6. By **S** Proposition 4.2.4(a), $\{A, \mathscr{T}_1\}$ and $\{A, \mathscr{T}_2\}$ are both Hausdorff spaces. Since $0 \in A$, A is closed in $\{\mathbf{R}, \mathscr{T}\}$ and hence $\{A, \mathscr{T}_2\}$ is a compact space. Of course, $\{A, \mathscr{T}_1\}$ is a compact space by the Heine–Borel Theorem. But the topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on A are non-comparable, for

$$\left[\frac{1}{2}, 1\right] \in \mathscr{T}_2 \text{ and } \left[\frac{1}{2}, 1\right] \notin \mathscr{T}_1 \text{ whilst } \left[0, \frac{1}{2}\right) \in \mathscr{T}_1 \text{ and } \left[0, \frac{1}{2}\right) \notin \mathscr{T}_2.$$

SAQ 14 Let T_1 be $[0,1]$ with the usual topology, and let T_2 be the subspace $[0,1]$ of $\mathbf{R}$ with the topology introduced in **S** Example 3.1.7, page 47. Show that the identity map on $[0,1]$ is a continuous bijection from T_1 to T_2 which is not a homeomorphism. Comment on this fact in relation to **S** Theorem 5.9.1.

SOLUTION TO SAQ 14 The topology of T_1 is strictly finer than the topology of T_2. To see this, we observe that a finite subset of $[0,1]$ is closed in T_1, so that each subset of $[0,1]$ which has a finite complement is open in T_1. Hence each set open in T_2 is open in T_1. Furthermore, $\left[0, \frac{1}{2}\right)$ is open in T_1 but not open in T_2, since its complement in $[0,1]$ is infinite. Thus the identity map on $[0,1]$ is a continuous map from T_1 to T_2, but it is not a continuous map from T_2 to T_1.

Whilst T_1 is a compact space, T_2 is not a Hausdorff space (see the argument in Note (iii), Section 5.5, *Unit* 5), so that **S** Theorem 5.9.1 does not apply. ■

6.5 Compact Subsets of Euclidean Spaces

In this section we obtain a complete description of the compact subsets of Euclidean spaces. It is based on the results of **S** Sections 5.3 and 5.4 and on the fact (**S** Theorem 5.6.2) that the topological product of compact spaces is compact. This is a substantial result, and its proof will require careful study.

READ S page 84, line -12 to the bottom of page 86.

NOTES

(i) *Page 86, Remark 5.6.3*
Do not be worried by the reference to an infinite product of spaces. We have not defined infinite Cartesian products, and infinite product spaces will play no part in our work.

(ii) *Page 86, Theorem 5.7.1*
In view of the results of **S** Section 5.4 (namely, Propositions 5.4.1 and 5.4.2), we can now state that a subset of $\mathbf{R}^n$ is compact *if and only if* it is closed and bounded.

It should be stressed that **S** Theorem 5.7.1 is a statement about $\mathbf{R}^n$ with the Euclidean metric. Thus 'bounded' means 'bounded in $\mathbf{R}^n$ with the Euclidean metric'.

(iii) *Page 86, line -10*
We must show that the bounded subset C of $\mathbf{R}^n$ is contained in a 'cube'

$$[a,b] \times [a,b] \times \cdots \times [a,b] \qquad (n \text{ factors})$$

for some a, b in $\mathbf{R}$. For this problem it is more convenient to use the metric d_∞ on $\mathbf{R}^n$ (see **S** Example 2.2.3) rather than the Euclidean metric d_2. The metrics d_∞ and d_2 on $\mathbf{R}^n$ are Lipschitz equivalent (**S** Example 2.4.5) and so C, being a bounded subset of $\{\mathbf{R}^n, d_2\}$, is a bounded subset of $\{\mathbf{R}^n, d_\infty\}$. Hence there exists a real number K such that $d_\infty(x, \mathbf{0}) \leq K$ for all x in C, where $\mathbf{0} = (0, 0, \ldots, 0)$. But if $x = (x_1, x_2, \ldots, x_n)$ then

$$d_\infty(x, \mathbf{0}) = \text{Max}\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Thus, if $x = (x_1, \ldots, x_n) \in C$ then $|x_i| \leq K$; so that $-K \leq x_i \leq K$ for $i = 1, \ldots, n$. Hence we obtain a cube containing C, as required, by taking $a = -K$ and $b = K$.

(iv) *Page 86, Remark 5.7.2*
This question has already been considered in Note (iv) on the first reading passage in Section 6.4. ▲

EXAMPLE 10 We have seen in Example 8 that there exist non-compact spaces on which every continuous real-valued function is bounded. We show now that we cannot obtain such an example which is a subspace of $\mathbf{R}$ with the usual topology; that is, we show that if a subset of A of $\mathbf{R}$ is not compact, then there exists a continuous function $f: A \to \mathbf{R}$ which is not bounded on A.

Let A be a subset of $\mathbf{R}$ which is not compact. Then, by **S** Theorem 5.7.1, either A is not bounded or A is not closed. If A is not bounded, then the real-valued function

$$f: x \mapsto x, \quad x \in A,$$

with domain A is continuous but not bounded on A.

If A is not closed, then $A \neq Cl(A)$ so that $Cl(A) - A \neq \varnothing$. Let $b \in Cl(A) - A$. Since $b \notin A$, it follows that $x - b$ is never zero for x in A; so that the real-valued function

$$f: x \mapsto \frac{1}{|x - b|}, \quad x \in A,$$

with domain A, is well-defined and continuous on A. Now since $b \in Cl(A)$ and $b \notin A$, we know that b is a limit point of A. Thus, for any given $\varepsilon > 0$, the open ball $B_\varepsilon(b)$ contains a point a of A; that is, there is an $a \in A$ such that $|a - b| < \varepsilon$, and hence

$$f(a) = \frac{1}{|a - b|} > \frac{1}{\varepsilon}.$$

Since ε is arbitrary, $\frac{1}{\varepsilon}$, and so also $f(a)$, can be as large as we please, so that f is unbounded on A.

Remark. In connection with **S** Corollary 5.5.4 (page 84), it can be shown that if A is a subspace of $\mathbf{R}$ which is not compact, then there exists a continuous function $f: A \to \mathbf{R}$ which is bounded but does not attain its bounds. ☐

Next we wish to present a famous example—the Cantor Ternary Set. One of the interesting properties of this example is new, that of being *perfect.*

DEFINITION A subset S of a topological space T is said to be *perfect in T* if S is closed in T and every point of S is a limit point of S.

EXAMPLE 11 We now look at some subsets of $\mathbf{R}^2$ (with the usual topology) with the above definition in mind.

(i) Let $S_1 = \{(x_1, x_2): x_1^2 + x_2^2 = 1.\}$ S_1 is closed in $\mathbf{R}^2$, and every point of S_1 is a limit point of S_1; hence S_1 is perfect.

(ii) Let S_2 be the singleton $\{(2, 1)\}$. S_2 is closed in $\mathbf{R}^2$. However, $(2, 1)$ is a point of S_2 that is *not* a limit point of S_2, so that S_2 is not perfect.

(iii) Let $S_3 = \mathbf{R}^2 - \{(2, 1)\}$. Every point of S_3 is a limit point of S_3. However, S_3 is not closed in $\mathbf{R}^2$, so that S_3 is not perfect. ☐

The following proposition gives more information about perfect sets. For a subset S of a topological space, T, let S' denote the set of limit points of S in T. We recall that $Cl(S) = S \cup S'$ (**S** Definition 3.7.11, page 61).

PROPOSITION 3 A subset S of a topological space T is perfect in T if and only if $S = S'$.

PROOF If S is perfect in T, then S is closed in T, so that $S' \subset S$. Also, every point of S is a limit point of S, so that $S \subset S'$. Hence $S = S'$.

Conversely, if $S = S'$, then $Cl(S) = S$ so that S is closed in T. Also, every point of S is a limit point of S, and hence S is perfect in T. ■

EXAMPLE 12 (The Cantor Ternary Set) There exists a subset K of $\mathbf{R}$ (with the usual topology) which is: (a) compact; (b) perfect in $\mathbf{R}$; and (c) nowhere-dense in $\mathbf{R}$.

We shall give a geometrical description of K. We do not have space to provide *every* detail of our arguments, so do not worry if you find it hard going. We hope you will be able to study our presentation closely enough to get a good overall impression of this intriguing example.

From the closed unit interval $[0, 1]$, delete the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ leaving

$$K_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

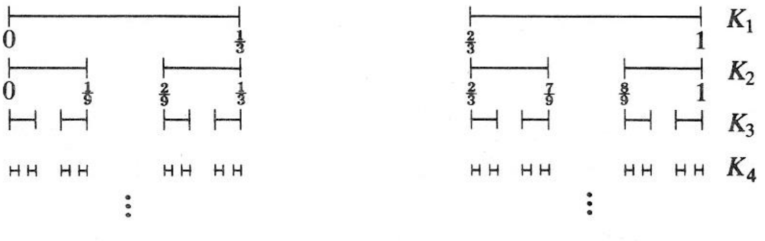
From each of the two closed intervals in K_1, delete the open middle third, leaving

$$K_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right],$$

the union of four closed intervals.

Now remove the open middle third of each of these four intervals, and continue this process indefinitely, at the nth stage obtaining a closed subset K_n of $[0, 1]$ which is the union of 2^n closed intervals each of length 3^{-n}, with $K_{n+1} \subset K_n$ for every n. The *Cantor Ternary Set* is defined to be

$$K = \bigcap_{n=1}^{\infty} K_n.$$



A general result (see SAQ 22(c)) implies that K is non-empty, but it is easy to see this directly. For if x is the end-point of one of the intervals of K_m, then x is not deleted at any subsequent stage of the construction, so that $x \in K_n$ for every $n > m$. Since $K_m \subset K_n$ if $n < m$, it follows that $x \in K_n$ for every positive integer n. Therefore $x \in K = \bigcap_{n=1}^{\infty} K_n$. In particular, $1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^n}, \ldots, 0$ are all points of K and so K is an infinite set.

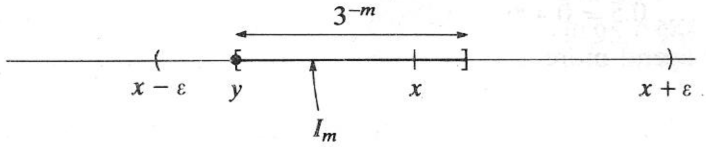
We can now justify our assertions about K.

(a) K is compact.

Since K is the intersection of the family $(K_n)_{n \in \mathbf{N}}$ of closed sets of $\mathbf{R}$, it is closed in $\mathbf{R}$. Since $K \subset [0, 1]$, it is bounded. Therefore K is a compact subset of $\mathbf{R}$.

(b) K is perfect in $\mathbf{R}$.

Since K is closed in $\mathbf{R}$, it remains to show that if $x \in K$, then x is a limit point of K. We must show that, if $\varepsilon > 0$ is given, then there exists y in K such that $y \neq x$ and $|y - x| < \varepsilon$. We achieve this by finding a positive integer m such that the interval of K_m which contains x has length less than ε.



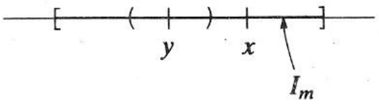
Thus we choose m such that $3^{-m} < \varepsilon$. Since $x \in K = \bigcap_{n=1}^{\infty} K_n$, it follows that $x \in K_m$. Therefore x belongs to a closed interval I_m of length 3^{-m} such that $I_m \subset K_m$. Let y be an end-point of I_m such that $y \neq x$. Then $y \in K$, as noted in our argument to show that K is non-empty. But since $x, y \in I_m$, we have

$$|y - x| \leq 3^{-m} < \varepsilon.$$

Thus x is a limit point of K.

(c) K is nowhere-dense in $\mathbf{R}$.

Since K is closed in $\mathbf{R}$, by S Corollary 3.7.29, page 64, it suffices to show that $\mathbf{R} - K$ is dense in $\mathbf{R}$. Thus we must show that each point of K is a limit point of $\mathbf{R} - K$. Let $x \in K$ and let $\varepsilon > 0$ be given. We must find y in $\mathbf{R} - K$ such that $|y - x| < \varepsilon$. Just as in the proof of part (b), we choose an integer m such that $3^{-m} < \varepsilon$ (so that K_m is a union of closed intervals of length less than ε). Since $x \in K_m$, there exists a closed interval I_m of length 3^{-m} such that $x \in I_m \subset K_m$. At the $(m + 1)$th stage of the construction, the open middle third of I_m is deleted. Thus we can choose y in I_m such that $y \notin K_{m+1}$.



Then $y \in \mathbf{R} - K$, and since $x, y \in I_m$ we have

$$|y - x| \leq 3^{-m} < \varepsilon.$$

Therefore x is a limit point of $\mathbf{R} - K$.

Now play the second part of the tape for Unit 6

SAQ 15 Which of the following subsets of $\mathbf{R}$ (with the usual topology) are compact?

(i) $\mathbf{Q} \cap [0, 1]$; (ii) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$; (iii) $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$;

(iv) $\{x \in \mathbf{R}: x^2 + x - 2 \geq 0\}$; (v) $\{x \in \mathbf{R}: x^2 + x - 2 \leq 0\}$.

SAQ 16 Which of the following subsets of $\mathbf{R}^2$ (with the usual topology) are compact?

(i) $\{(x_1, x_2) \in \mathbf{R}^2: x_1^2 + x_2^2 < 1\}$
(ii) $\{(x_1, x_2) \in \mathbf{R}^2: x_1^2 + x_2^2 \leq 1\}$
(iii) $S^1 = \{(x_1, x_2) \in \mathbf{R}^2: x_1^2 + x_2^2 = 1\}$
(iv) $\{(x_1, x_2) \in \mathbf{R}^2: x_1^2 + x_2^2 > 1\}$

SAQ 17 Let A and B be compact subsets of $\mathbf{R}$ (with the usual topology).
Prove that

$$\{x + y: x \in A, y \in B\}$$

is a compact subset of $\mathbf{R}$. [Hint: Recall the result of SAQ 25(i) on page 112 of *Unit 4*.]

SAQ 18 Which of the following subsets of $\mathbf{R}$ are perfect in $\mathbf{R}$ (with the usual topology)?

(i) $[0, 1)$; (ii) $[0, 1]$; (iii) $[0, 1] \cup \{2\}$; (iv) $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

***SAQ 19** Let D be the subset of $[0,1]$ which consists of all real numbers x which have a decimal expansion

$$x = 0.x_1x_2x_3\ldots$$

where $x_i \neq 5$ for every i. (In such a representation we replace any terminating decimal by the equivalent representation ending in infinitely many 9s: e.g. $1 = 0.99\ldots$, $0.5 = 0.499\ldots$.) Prove that D with the usual topology is compact.

(Do not spend more than 10 minutes on this SAQ before reading its solution.)

SOLUTION TO SAQ 15

(i) $\mathbf{Q} \cap [0,1]$ is bounded but not closed so it is not compact.

(ii) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is bounded but not closed (as it does not contain its limit point 0), so it is not compact.

(iii) $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is both bounded and closed so it is compact. (It does contain its limit points.)

(iv) Let $E = \{x \in \mathbf{R}: x^2 + x - 2 \geq 0\}$. Since

$$x^2 + x - 2 = (x+2)(x-1),$$

we see that $x^2 + x - 2 \geq 0$ if and only if either $x \leq -2$ or $x \geq 1$. Thus

$$E = (-\infty, -2] \cup [1, \infty),$$

so that E is closed but not bounded. Therefore E is not compact.

(v) By a similar argument to that employed in part (iv), we see that

$$\{x \in \mathbf{R}: x^2 + x - 2 \leq 0\} = [-2, 1],$$

and so it is compact. ■

SOLUTION TO SAQ 16

(i) This set is not closed in $\mathbf{R}^2$ and so is not compact.

(ii) This set is closed and bounded and hence compact.

(iii) S^1 is closed and bounded and hence compact.

(iv) This set is not bounded in $\mathbf{R}^2$ and so is not compact. ■

SOLUTION TO SAQ 17 Let

$$C = \{x + y: x \in A, y \in B\}.$$

It was shown in SAQ 25(i) on page 112 of *Unit 4* that $\sigma: \mathbf{R}^2 \to \mathbf{R}$ given by

$$\sigma(x, y) = x + y, \quad (x, y) \in \mathbf{R}^2,$$

is continuous. Now $A \times B \subset \mathbf{R}^2$ and

$$\sigma(A \times B) = C.$$

Since A and B are compact, it follows from **S** Theorem 5.6.2 that $A \times B$ is compact. Thus C is the continuous image of a compact space and so C is compact (**S** Proposition 5.5.1). ■

SOLUTION TO SAQ 18

(i) $[0, 1)$ is not closed in $\mathbf{R}$ and so is not perfect.

(ii) $[0, 1]$ is perfect in $\mathbf{R}$.

(iii) $S = [0, 1] \cup \{2\}$ is closed in $\mathbf{R}$, but it is not perfect since 2 is not a limit point of S.

(iv) $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is closed in $\mathbf{R}$, but it is not perfect since no point of S other than 0 is a limit point of S. ■

SOLUTION TO SAQ 19 Let D_1 be the subset of $[0,1]$ consisting of those real numbers x which have a decimal expansion $x = 0.x_1x_2\ldots$ with $x_1 \neq 5$.

Then

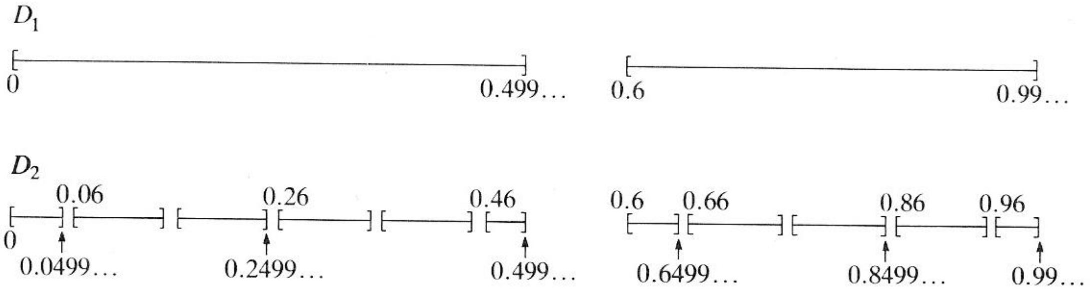
$$D_1 = [0, 0.5] \cup [0.6, 1].$$

Note that $0.5 \in D_1$ and $1 \in D_1$, since 0.5 and 1 have equivalent representations $0.499\ldots$ and $0.99\ldots$ respectively.

Thus D_1 is a closed subset of $\mathbf{R}$ which is the union of 9 closed intervals each of length $\frac{1}{10}$.

Now let D_2 be the subset of $[0, 1]$ consisting of those real numbers x which have a decimal expansion $x = 0.x_1x_2x_3\ldots$ where $x_1 \neq 5$ and $x_2 \neq 5$. Then

$$D_2 = D_1 - ((0.05, 0.06) \cup (0.15, 0.16) \cup \cdots \cup (0.95, 0.96)).$$



Therefore D_2 is a closed subset of $\mathbf{R}$ which is the union of 9^2 closed intervals each of length 10^{-2}.

In general, let D_n be the subset of $[0, 1]$ consisting of those real numbers x which have a decimal expansion $x = 0.x_1x_2\ldots x_nx_{n+1}\ldots$ where $x_i \neq 5$ for $i = 1, \ldots, n$. Then D_n is a closed subset of $\mathbf{R}$ (being the union of 9^n closed intervals of length 10^{-n}). Also, $D_{n+1} \subset D_n$ for every n. We then see that

$$D = \bigcap_{n=1}^{\infty} D_n.$$

Thus D, being the intersection of sets closed in $\mathbf{R}$, is closed in $\mathbf{R}$. Also, $D \subset [0, 1]$, so that D is bounded. Therefore D is compact. ■

6.6 Uniform Continuity

In this section we return to the study of metric spaces, and consider the concept of uniform continuity of functions between metric spaces. We shall see that, in general, a continuous function need not be uniformly continuous; but that, for functions whose domain is compact, continuity and uniform continuity are equivalent concepts.

READ S all of page 87.

NOTES (i) *Page 87, Proposition 5.8.2*

The proof of this is rather subtle; to help you, we explain S's use of $2\delta(x)$ in the second line of the proof.

Since f is continuous at x, by applying **S** Definition 2.1.3 (page 22) with the positive number $\frac{1}{2}\varepsilon$ playing the role of 'ε', we find that there exists $\delta > 0$ (depending on x and ε) such that $d_2(f(x), f(y)) < \frac{1}{2}\varepsilon$ for all y in M_1 satisfying $d_1(x, y) < \delta$. Now put $\delta(x) = \frac{1}{2}\delta$. Then $d_2(f(x), f(y)) < \frac{1}{2}\varepsilon$ for all y in M_1 satisfying $d_1(x, y) < 2\delta(x)$.

(ii) *Page 88, lines 1–15*

We are asking you to read these definitions now, even though we shall not refer to them again in this unit. They will be referred to in *Unit 8*. ▲

EXAMPLE 13 Let $A = (0,1)$, and consider the functions $f, g: A \to \mathbf{R}$ given by

$$f(x) = x^2, \quad g(x) = \frac{1}{x}.$$

Let us use the usual Euclidean metric on $\mathbf{R}$, and let A have the induced metric as a subspace of $\mathbf{R}$.

We shall prove that f is uniformly continuous on A, whilst g is not uniformly continuous on A.

To show that f is uniformly continuous, we must consider

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y|\,|x + y|$$

for $0 < x, y < 1$. Now $|x + y| < 2$, so that

$$|f(x) - f(y)| < 2|x - y|.$$

Thus, given $\varepsilon > 0$, let us take $\delta = \frac{1}{2}\varepsilon$. If $x, y \in A$ and $|x - y| < \delta$ then

$$|f(x) - f(y)| < 2\delta = \varepsilon.$$

Therefore f is uniformly continuous.

We know that g is continuous, but we shall prove that it is not uniformly continuous. We should expect this to be the case, for if x, $x + a$ are elements of the domain which differ by a (a fixed number), then $g(x) - g(x + a)$ increases without bound as $x \longrightarrow 0$. So we aim for a contradiction argument. Suppose that g is uniformly continuous. Then, taking $\varepsilon = 1$ in **S** Definition 5.8.1, there exists $\delta > 0$ such that

$$|g(x) - g(y)| < 1$$

for all $x, y \in A$ such that $|x - y| < \delta$, and there is no loss of generality in supposing that $\delta < 1$. Now let $x_0 = \frac{1}{2}\delta$, $y_0 = \delta$. Then $|x_0 - y_0| = \frac{1}{2}\delta < \delta$, but

$$|g(x_0) - g(y_0)| = \left|\frac{1}{x_0} - \frac{1}{y_0}\right| = \frac{|y_0 - x_0|}{|x_0 y_0|} = \frac{\frac{1}{2}\delta}{\frac{1}{2}\delta^2} = \frac{1}{\delta} > 1.$$

This contradiction shows that g is not uniformly continuous.

SAQ 20 Determine whether in terms of the usual metrics on A and $\mathbf{R}$, the function $f: A \to \mathbf{R}$ is uniformly continuous on A, when:

(a) $A = (0,1)$, $f(x) = \dfrac{1}{1 - x}$;

(b) $A = [1, \infty)$, $f(x) = \dfrac{1}{x}$.

SAQ 21 Let M be a metric space with metric d and let a be a point of M. Show that the real-valued function

$$x \mapsto d(x, a), \quad x \in M,$$

is uniformly continuous on M.

[Hint: Recall the result of SAQ 4 on page 71 of *Unit 3*.]

SOLUTION TO SAQ 20 (a) By analogy with the function $x \mapsto \dfrac{1}{x}$ on A, considered in Example 13, we should expect that f is not uniformly continuous on A. We can use a similar argument, except that, in this case, we expect the uniform continuity condition to break down near 1 rather than near 0.

Thus we suppose that f is uniformly continuous on A. Then (taking $\varepsilon = 1$ in **S** Definition 5.8.1) there exists δ such that $0 < \delta < 1$ and

$$|f(x) - f(y)| < 1$$

for all x, y in A satisfying $|x - y| < \delta$. Now let $x_0 = 1 - \delta$, $y_0 = 1 - \frac{1}{2}\delta$. Then $x_0, y_0 \in A$ and $|x_0 - y_0| < \delta$. But

$$|f(x_0) - f(y_0)| = \left|\left(1 - \left(1 - \frac{1}{2}\delta\right)\right)^{-1} - (1 - (1 - \delta))^{-1}\right| = \frac{1}{\delta} > 1.$$

This contradiction shows that f is not uniformly continuous on A.

(b) If $x, y \in A$ then

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{|xy|}.$$

But $x \geq 1$ and $y \geq 1$ so that $|xy| \geq 1$ and hence $\frac{1}{|xy|} \leq 1$. Therefore

$$|f(x) - f(y)| \leq |x - y| \quad \text{for all } x, y \in A.$$

Thus given $\varepsilon > 0$, we take $\delta = \varepsilon$; then

$$|f(x) - f(y)| < \varepsilon$$

for all x, y in A satisfying $|x - y| < \delta$. Therefore f is uniformly continuous on A. ■

SOLUTION TO SAQ 21 It was shown in SAQ 4 on page 71 of *Unit 3* that

$$|d(x, a) - d(y, a)| \leq d(x, y)$$

for all $x, y \in M$. Thus, given $\varepsilon > 0$, taking $\delta = \varepsilon$ we have

$$|d(x, a) - d(y, a)| < \varepsilon$$

for all x, y in M satisfying $d(x, y) < \delta$. Therefore the given real-valued function is uniformly continuous on M. ■

6.7 Objectives of Unit 6

After studying the unit, you should:

(1) know the definition of compactness, and be able to determine (using open covers) whether or not a given space is compact;

(2) know the Heine–Borel and Bolzano–Weierstrass Theorems;

(3) know the connections between compactness, boundedness and closure of various types of spaces;

(4) know the properties of continuous functions on compact spaces;

(5) know the inverse function theorem concerning a continuous one-one map of a compact space onto a Hausdorff space;

(6) know that the topological product of two compact topological spaces is compact;

(7) know the Heine–Borel Theorem for $\mathbf{R}^n$;

(8) know the definition of a perfect set, and be able to determine whether or not a given set is perfect;

(9) understand the geometric and the number-theoretic constructions of the Cantor Ternary Set, and why it is compact, perfect in $\mathbf{R}$ and nowhere-dense in $\mathbf{R}$, and uncountable;

(10) know the definition of uniform continuity of a map between metric spaces, and be able to determine whether or not a given map is uniformly continuous;

(11) know that a continuous map between metric spaces whose domain is compact is uniformly continuous.

6.8 Revision Questions

SAQ 22 (a) Show that if $(E_i)_{i\in I}$ is a family of closed sets of a topological space T, then $(T-E_i)_{i\in I}$ is an open cover of T if and only if $\bigcap\limits_{i\in I} E_i = \varnothing$.

(b) Prove that a compact space T satisfies the following condition: $\bigcap\limits_{i\in I} E_i \neq \varnothing$ for each family $(E_i)_{i\in I}$ of closed sets of T with the property that $\bigcap\limits_{i\in J} E_i \neq \varnothing$ for every finite subset J of I.

(c) Prove that if $(F_n)_{n\in \mathbf{N}}$ is a family of non-empty closed sets of a compact space such that $F_{n+1} \subset F_n$ for every n, then $\bigcap\limits_{n=1}^{\infty} F_n \neq \varnothing$.

SAQ 23 Let $A = [-1, 1]$ and let $\mathscr{T}$ be the collection of subsets of A which consists of A, $[-1, 1)$, $(-1, 1]$, $(-1, 1)$ and all subsets of A which do not contain 0.

(a) Prove that $\mathscr{T}$ is a topology for A.

(b) Show that $T = \{A, \mathscr{T}\}$ is a compact space.

*(c) Show that a subset B of T is compact if and only if it contains 0 or is finite.

***SAQ 24** S Exercise 5.10.16, page 90.

SAQ 25 Show that if H and K are compact subsets of a Hausdorff space T, then $H \cap K$ is compact.

***SAQ 26** Let S and T be topological spaces which have the property that a subspace of the topological product $S \times T$ is compact if and only if it is closed. Prove that a subspace of S is compact if and only if it is closed.

SAQ 27 Prove that if $f: M_1 \to M_2$ and $g: M_2 \to M_3$ are uniformly continuous maps of metric spaces, then $g \circ f: M_1 \to M_3$ is uniformly continuous.

SOLUTION TO SAQ 22 (a) Since E_i is closed in T, $(T-E_i)_{i\in I}$ is a family of open sets of T. It is an open cover of T if and only if

$$\bigcup_{i\in I}(T-E_i) = T.$$

But by Proposition 3 (De Morgan's Laws) on page 16 of *Unit 1*,

$$\bigcup_{i\in I}(T-E_i) = T - \left(\bigcap_{i\in I} E_i\right).$$

Thus $(T-E_i)_{i\in I}$ is an open cover of T if and only if $\bigcap\limits_{i\in I} E_i = \varnothing$.

(b) Let T be a compact space and let $(E_i)_{i\in I}$ be a family of closed sets of T with the property that $\bigcap\limits_{i\in J} E_i \neq \varnothing$ for every finite subset J of I. We must show that $\bigcap\limits_{i\in I} E_i \neq \varnothing$. Suppose that this is not the case, so that $\bigcap\limits_{i\in I} E_i = \varnothing$. Then by part (a), $(T-E_i)_{i\in I}$ is an open cover of the compact space T and so has a finite subcover. Thus there exists a finite subset J of I such that $(T-E_i)_{i\in J}$ is a cover of T. Then it follows from part (a) that $\bigcap\limits_{i\in J} E_i = \varnothing$, which is a contradiction. Therefore $\bigcap\limits_{i\in I} E_i \neq \varnothing$.

(c) This is an immediate consequence of part (b). For let J be a finite subset of $\mathbf{N}$ and let m be the largest member of J. Then $\bigcap\limits_{i\in J} F_i = F_m \neq \varnothing$. Thus $\bigcap\limits_{i\in J} F_i \neq \varnothing$ for every finite subset J of $\mathbf{N}$. Hence $\bigcap\limits_{n=1}^{\infty} F_n \neq \varnothing$. ■

SOLUTION TO SAQ 23 (a) $A \in \mathscr{T}$ by definition, and $\varnothing \in \mathscr{T}$ since $0 \notin \varnothing$. Thus (T1) is satisfied.

To verify (T2), let $V, W \in \mathscr{T}$. If either $0 \notin V$ or $0 \notin W$ then $0 \notin V \cap W$; if this is not the case then $V \supset (-1, 1)$ and $W \supset (-1, 1)$ so that $V \cap W \supset (-1, 1)$.

Therefore $V \cap W \in \mathcal{T}$. Finally, let $(U_i)_{i\in I}$ be a family of members of $\mathcal{T}$ and let $U = \bigcup_{i\in I} U_i$. To verify (T3) we must show that $U \in \mathcal{T}$. If there is some j in I such that $0 \in U_j$ then $U_j \supset (-1,1)$, so that $U \supset (-1,1)$ and hence $U \in \mathcal{T}$. On the other hand, if, for every i, $0 \notin U_i$ then $0 \notin U$ and so again $U \in \mathcal{T}$. This completes the verification of (T3).

Hence $\mathcal{T}$ is a topology for A.

(b) Let $\mathcal{U}$ be an open cover of $\mathcal{T}$. Then there exists a U_0 in $\mathcal{U}$ such that $0 \in U_0$. But the only open sets of T which contain 0 are $(-1,1)$, $(-1,1]$, $[-1,1)$ and A. Let U_1 be a member of $\mathcal{U}$ which contains -1 and U_2 be a member of $\mathcal{U}$ which contains 1.

Then $\{U_0, U_1, U_2\}$ is a finite subcover of $\mathcal{U}$. Hence T is a compact space.

(c) Since T is compact, every closed set of T is compact.

(These sets are A and $\varnothing$, $\{1\}$, $\{-1\}$, $\{-1,1\}$ and all subsets of A that contain 0.)

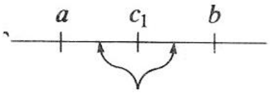
It remains to determine which non-closed sets of T are compact. Let B be a subset of T which is not closed. Then $0 \notin B$, since every set which contains 0 is closed. We can now see that the induced topology on B is the discrete topology. For if $V \subset B$ then $0 \notin V$ so that V is open in T; hence every subset of B is open. It follows that B is compact if and only if it is finite.

Hence a subset of T is compact if and only if it is either closed or finite; or, equivalently, if it contains 0 or is finite. ■

OLUTION TO SAQ 24

We employ the argument suggested by **S** (page 90, lines -15 to -4).

Let $\mathcal{U}$ be a cover of $[a,b]$ by sets open in $\mathbf{R}$. Assume that there is no finite subcover of $\mathcal{U}$ and put $c_1 = \frac{1}{2}(a+b)$. Then at least one of $[a,c_1]$ and $[c_1,b]$ is not covered by any finite subcollection of $\mathcal{U}$.



Choose one of these, and call it $[a_1, b_1]$

Let $[a_1,b_1]$ denote one of these intervals, not covered by a finite subcollection of $\mathcal{U}$. We note that $a \le a_1 < b_1 \le b$ and $b_1 - a_1 = \frac{1}{2}(b-a)$. Now repeat the argument, by bisecting $[a_1,b_1]$ to obtain $[a_2,b_2]$ not covered by any finite subcollection of $\mathcal{U}$, and so on.

By induction, we obtain two real sequences (a_n), (b_n) with $a_n \le a_{n+1} < b_{n+1} \le b_n$ for all n, $b_n - a_n = (b-a)/2^n$, and such that $[a_n,b_n]$ is not covered by any finite subcollection of $\mathcal{U}$. Notice that for each positive integer k, we have $a_n < b_k$ for every n. Thus the increasing sequence (a_n) is bounded above and so converges, to c (say), by **S** Theorem 1.2.6 on page 9. Now since $c = \sup\{a_n\}$, we have $c \le b_k$ for every positive integer k, so that, by **S** Theorem 1.2.6 again, the decreasing sequence (b_n) converges, to d (say). Since $c \le b_k$ for every k and $d = \inf\{b_k\}$, we also deduce that $c \le d$.

Also

$$a_n \le c \le d \le b_n$$

so that

$$0 \le d - c \le b_n - a_n = \frac{b-a}{2^n}$$

for every n. Since $(b-a)/2^n \longrightarrow 0$ as $n \longrightarrow \infty$ we deduce that $d = c$.

Now $c \in [a,b]$ and $\mathcal{U}$ is a cover of $[a,b]$, so that there exists U in $\mathcal{U}$ such that $c \in U$. Since U is open in $\mathbf{R}$, there exists $\varepsilon > 0$ such that $B_\varepsilon(c) \subset U$. Thus

$(c - \varepsilon, c + \varepsilon) \subset U$. We can now obtain a contradiction by noting that there exists a positive integer n such that

$$b_n - a_n = \frac{b - a}{2^n} < \varepsilon.$$

Since $c \in [a_n, b_n]$, it follows that

$$[a_n, b_n] \subset (c - \varepsilon, c + \varepsilon) \subset U,$$

contrary to our hypothesis that $[a_n, b_n]$ is not covered by any finite subcollection of U.

This contradiction shows that some finite subcollection of $\mathscr{U}$ covers $[a, b]$. Thus we have found another proof that $[a, b]$ is compact. ■

SOLUTION TO SAQ 25 Since K is a compact subset of the Hausdorff space T, it is closed in T by S Proposition 5.4.2. Thus $H \cap K$ is closed in the compact subspace H, and so $H \cap K$ is compact by S Proposition 5.6.1. ■

Remark (for interest only). The intersection of compact subsets of a non-Hausdorff space need *not* be compact.

Let D be the indiscrete space whose elements are 0 and 1 and consider the topological product $T = \mathbf{R} \times D$ where $\mathbf{R}$ has the usual topology. Note that non-empty open sets of T are of the form $U \times D$ where U is an open set of $\mathbf{R}$.

Now let

$$H = ([0,1] \times \{0\}) \cup ((0,1) \times \{1\}) \text{ and}$$
$$K = ((0,1) \times \{0\}) \cup ([0,1] \times \{1\}).$$

We show first that H is a compact subset of T. Any cover $\mathscr{V}$ of H by sets open in T consists of sets $U \times D$ where the U's are open in $\mathbf{R}$ and form a cover $\mathscr{U}$ of $[0,1]$. Since $[0,1]$ is compact, there is a finite subcover $\{U_1, \dots, U_k\}$ of $\mathscr{U}$ for $[0,1]$. Then $\{U_1 \times D, \dots, U_k \times D\}$ is a finite subcover of $\mathscr{V}$ for H. Hence H is compact.

By a similar argument, K is compact.

But

$$H \cap K = (0,1) \times D,$$

which is not compact. For example, if $V_n = \left(\frac{1}{n}, 1\right) \times D$, then $(V_n)_{n \in \mathbf{N}}$ is an open cover of $H \cap K$ which has no finite subcover.

SOLUTION TO SAQ 26 Note that since $S \times T$ is closed in $S \times T$, by hypothesis $S \times T$ is compact. It follows from S Theorem 5.6.2 that S and T are compact.

Now let B be a subset of S. If B is closed in S then B is compact, by S Proposition 5.6.1.

Conversely, if B is compact, then $B \times T$ is a compact subset of $S \times T$ by S Theorem 5.6.2. Hence, by hypothesis, $B \times T$ is closed in $S \times T$, from which it follows that B is closed in S. Therefore B is compact if and only if it is closed, as required. ■

SOLUTION TO SAQ 27 Suppose that the metric on M_i is called d_i for $i = 1, 2, 3$.

Let $\varepsilon > 0$ be given. Since g is uniformly continuous, there exists $\eta > 0$ such that

$$d_3(g(z), g(w)) < \varepsilon$$

for all z, w in M_2 such that $d_2(z, w) < \eta$. Also, since f is uniformly continuous, there exists $\delta > 0$ such that

$$d_2(f(x), f(y)) < \eta$$

for all x, y in M_1 such that $d_1(x, y) < \delta$.

It follows that if $x, y \in M_1$ and $d_1(x, y) < \delta$, then

$$d_3(g(f(x)), g(f(y))) < \varepsilon.$$

But $g(f(x)) = (g \circ f)(x)$, so that

$$d_3((g \circ f)(x), (g \circ f)(y)) < \varepsilon$$

for all x, y in M_1 satisfying $d_1(x, y) < \delta$. Hence $g \circ f$ is uniformly continuous. ■

UNIT 7

CONNECTEDNESS

7.0 Introduction

In *Unit 6* we considered compactness as a generalization of a basic property of bounded closed intervals in **R**. Connectedness is a generalization of a property of all intervals in **R**, namely that of being all in 'one piece'. So, loosely, a space is connected if it does not consist of two or more separate pieces.

The formal definition of *connectedness* is studied in Section 7.1. Not surprisingly, a subspace of **R** with the usual topology is connected if and only if it is an interval.

For a subspace of Euclidean space, $\mathbf{R}^n$, there is another notion of 'connectedness' which may seem more natural: namely, the property that each pair of its points can be joined by a path in the subspace. *Path-connectedness*, which results from formalization of this idea, is studied in Section 7.3. We shall see that each path-connected space is connected, whilst there exist connected spaces which are not path-connected.

Section 7.4 contains results about connectedness which are useful for distinguishing between non-homeomorphic spaces.

7.1 The Definition of Connectedness

READ S from page 93, line −6 to page 97, line 5.

NOTES (i) *Page 94, Definition 6.2.1*

As indicated by the discussion at the top of page 94, we usually take the 'discrete two-point space' to be $D = \{0, 1\}$. Notice also that the continuous map must be *onto*, since for *any* topological space T there is a continuous map from T into D given by

$$f : x \mapsto 0 \qquad (x \in T).$$

(ii) *Page 94, line −12*

Definition 6.2.2 tells us that A and B are both open in T. It follows from Definition 3.7.1, **S** page 59, that the complements

$$T - A = B, \quad T - B = A$$

of these sets are closed in T.

(iii) *Page 95, lines 4–6*

(In this chapter, **S** uses the abbreviated notation $f^{-1}(c)$ instead of the more precise $f^{-1}(\{c\})$, for various $c \in \mathbf{R}$.) It must be shown that if f is a continuous map of a space T *onto* the discrete space $D = \{0, 1\}$, then $f^{-1}(0) \,|\, f^{-1}(1)$ is a partition of T.

First, we note that $f^{-1}(0)$ and $f^{-1}(1)$ are non-empty, since f is onto. Also, if $x \in T$ then either $f(x) = 0$ or $f(x) = 1$, so that x belongs to precisely one of $f^{-1}(0)$ and $f^{-1}(1)$. Hence

$$f^{-1}(0) \cup f^{-1}(1) = T \quad \text{and} \quad f^{-1}(0) \cap f^{-1}(1) = \varnothing.$$

Finally, $\{0\}$ and $\{1\}$ are open in the discrete space D, so continuity implies that $f^{-1}(0)$ and $f^{-1}(1)$ are open in T. Thus $f^{-1}(0) \,|\, f^{-1}(1)$ is a partition of T.

(iv) *Page 95, line 14*

In other words, the following are equivalent criteria for a topological space T to be connected:

(1) there does not exist any continuous map from T onto a discrete two-point space;

(2) T admits no partition;

(3) the only subsets of T which are both open and closed in T are $\varnothing$ and T.

Henceforth we shall use the three characterizations interchangeably, without further reference.

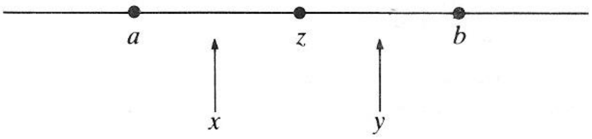
(v) *Page 95, Proposition 6.2.5*
In Proposition 1 below, we shall prove a result which is more general than S Proposition 6.2.5.

(vi) *Page 95, line -2 to page 96, line 2*
To illustrate S's temporary convention, consider the case in which the given subset S is bounded above but not bounded below. Then the assertion is that $(-\infty, b) \subset S \subset [-\infty, b]$, which is to be interpreted as

$$(-\infty, b) \subset S \subset (-\infty, b].$$

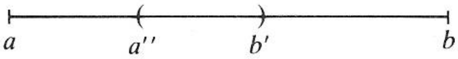
Note that, when this assertion has been verified, then we know that either $S = (-\infty, b)$ or $S = (-\infty, b]$.

(vii) *Page 96, lines 3 and 4*



Let us amplify the argument to show that if $z > a$ then there exists x in S with $x < z$. If S is bounded below, then the existence of x follows from the definition of a as $\inf S$. On the other hand, if S is not bounded below, then the existence of x follows from the fact that z is not a lower bound of S.

(viii) *Page 96, proof of Theorem 6.2.8*
The essence of this proof is that the assumption that there exists a partition of I leads us to conclude that there exists a *non-empty* interval (a'', b') contained in a closed interval $[a, b]$ but disjoint from each of two sets whose union is $[a, b]$.



This contradiction shows that I admits no partition and hence that I is connected. ▲

ROPOSITION 1 If A and B are closed sets of a topological space T such that $A \cup B = T$, and if $f: T \to T'$ is a map to a topological space T' such that $f|A$ and $f|B$ are continuous, then f is a continuous map.

PROOF By S Proposition 3.7.8 (page 61), it is enough to show that $f^{-1}(E)$ is a closed set of T for each closed set E of T'.

Since $A \cup B = T$, it follows that

$$\begin{aligned} f^{-1}(E) &= f^{-1}(E) \cap (A \cup B) \\ &= (f^{-1}(E) \cap A) \cup (f^{-1}(E) \cap B) \end{aligned}$$

(by Example 5(i) on page 11 of *Unit 1*).

Now, by Proposition 6 on page 20 of *Unit 1*,

$$f^{-1}(E) \cap A = (f \mid A)^{-1}(E).$$

Since $f \mid A: A \to T'$ is continuous, it follows from S Proposition 3.7.8 that $(f \mid A)^{-1}(E)$ is closed in A. But A is closed in T, so that $(f \mid A)^{-1}(E)$ is closed in T. Thus $f^{-1}(E) \cap A$ is closed in T. Similarly, $f^{-1}(E) \cap B$ is closed in T.

Therefore, $f^{-1}(E)$ is the union of two sets each of which is closed in T, and so $f^{-1}(E)$ is closed in T. Hence $f: T \to T'$ is continuous. ■

Remark. **S** Proposition 6.2.5 *is* a special case of Proposition 1; for, if $A|B$ is a partition of T, then A and B are closed sets of T such that $A \cup B = T$.

EXAMPLE 1 Consider the topological space $T = \{A, \mathscr{T}\}$ where $A = \{a, b\}$ and $\mathscr{T} = \{\varnothing, \{a\}, A\}$.

The closed sets of T are A, $\{b\}$, and $\varnothing$. Thus the only subsets which are both open and closed in T are $\varnothing$ and A. It follows that T is a connected space.

EXAMPLE 2 Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the topology for $\mathbf{R}$ introduced in **S** Example 3.1.7, page 47. (Recall that the open sets of T are $\varnothing$ and all subsets V of $\mathbf{R}$ such that $\mathbf{R} - V$ is finite.) We shall show that T is a connected space.

Suppose that T is *not* connected. Then there exists some partition $A|B$ of T. But then A and B are non-empty disjoint open sets of T. This is a contradiction, for in Note (iv) of Section 4.1 of *Unit 4* (page 95) it was shown that every two non-empty open sets of T have a non-empty intersection. Therefore T is a connected space.

EXAMPLE 3 (a) The subspace $T = (0, 1) \cup (10, 11)$ of $\mathbf{R}$ (with the usual topology) is not connected. For example, if we put $A = (0, 1)$ and $B = (10, 11)$ then $A|B$ is a partition of T, so that T is not connected.

(b) The subspace $\mathbf{Q}$ of $\mathbf{R}$ (with the usual topology) is not connected. For since $\sqrt{2}$ is an irrational number, if we put $A = (-\infty, \sqrt{2}) \cap \mathbf{Q}$ and $B = (\sqrt{2}, +\infty) \cap \mathbf{Q}$, then $A|B$ is a partition of $\mathbf{Q}$.

Finally, notice that in both (a) and (b) we could have used Theorems 6.2.7 and 6.2.8 rather than Proposition 6.2.3 to show that neither T nor $\mathbf{Q}$ could be connected—for the two theorems show that the connected subspaces of $\mathbf{R}$ are intervals, and neither T nor $\mathbf{Q}$ is an interval.

EXAMPLE 4 Let us look again at one of our 'old friends', and we shall see that it is another example of a topological space which is not connected.

Let $T = \{\mathbf{R}, \mathscr{T}\}$ where $\mathscr{T}$ is the topology for $\mathbf{R}$ introduced in Example 3 on page 96 of *Unit 4*. (Recall that a subset V of $\mathbf{R}$ is open in T if and only if either $0 \notin V$ or $\mathbf{R} - V$ is finite.)

We show that T is not connected by exhibiting a partition of T. Let $A = \{1\}$ and $B = \mathbf{R} - \{1\}$. Then A is open in T since $0 \notin A$; and B is open in T since $\mathbf{R} - B = \{1\}$, which is finite. Since A and B are disjoint and $A \cup B = T$, it follows that $A|B$ is a partition of T. Hence T is not a connected space.

Remark. In Example 3 of *Unit* 5, it was noted that every finite subset of T which does not contain 0 is both open and closed in T. Thus T has subsets, different from $\varnothing$ and T, which are both open and closed in T. This gives us an alternative proof that T is not connected, based on **S** Corollary 6.2.4.

SAQ 1 Show that

(i) a discrete space (defined in Example 3.1.4, **S** page 46) having more than one point is not connected; and

(ii) an indiscrete space (defined in Example 3.1.6, **S** page 47) is connected.

SAQ 2 Show that there are precisely 10 topologies $\mathscr{T}$ on the set $A = \{a, b, c\}$ such that $\{A, \mathscr{T}\}$ is not a connected space.

[Hint: In the Solution to SAQ 2(ii) of *Unit 4*, it was established that the topologies on A are of 9 types; check one representative of each type.]

SAQ 3 Let $\mathscr{T}_1, \mathscr{T}_2$ be topologies on a set A such that $\mathscr{T}_1 \subset \mathscr{T}_2$ and let $T_i = \{A, \mathscr{T}_i\}$, $i = 1, 2$. Prove that if T_2 is connected then T_1 is connected. Is it true that if T_1 is connected then T_2 is necessarily connected? [If *yes*, prove your assertion; if *no*, give a counter-example.]

SAQ 4 Prove that a topological space T is connected if and only if the boundary, $b(H)$, of every non-empty proper subset H of T is non-empty.
[Hint: Recall the result of Proposition 7 on page 15 of *Unit 5*.]

SAQ 5 Prove that a topological space T is connected if and only if T has the following property: for every discrete space T_1, every continuous map of T into T_1 is constant.

SOLUTION TO SAQ 1

(i) Let T_1 be a discrete space with more than one point, and let $a \in T_1$. Then $\{a\}$ and $T_1 - \{a\}$ are non-empty sets both of which are open in T_1 (since every subset is open in T_1). Thus $\{a\} \mid T_1 - \{a\}$ is a partition of T_1, and hence T_1 is not connected.

(ii) On the other hand, if T_2 is an indiscrete space, then $\varnothing$ and T_2 are the only open sets of T_2; and so T_2 is a connected space. ■

SOLUTION TO SAQ 2

In the Solution to SAQ 2(ii) on page 98 of *Unit 4*, it was shown that every topology on A, other than the discrete and indiscrete topologies, is the same type as one of the following topologies:

$$\begin{aligned}
\mathscr{T}_1 &= \{\varnothing, \{a\}, A\},\\
\mathscr{T}_2 &= \{\varnothing, \{a,b\}, A\},\\
\mathscr{T}_3 &= \{\varnothing, \{a\}, \{a,b\}, A\},\\
\mathscr{T}_4 &= \{\varnothing, \{a\}, \{b,c\}, A\},\\
\mathscr{T}_5 &= \{\varnothing, \{a\}, \{b\}, \{a,b\}, A\},\\
\mathscr{T}_6 &= \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, A\},\\
\mathscr{T}_7 &= \{\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}, A\}.
\end{aligned}$$

By inspection, if $i \neq 4, 7$ then $\{A, \mathscr{T}_i\}$ admits no partition (check each one!); and so each $\{A, \mathscr{T}_i\}$ is a connected space.

Next, notice that $\{A, \mathscr{T}_4\}$ has a partition $\{a\} \mid \{b,c\}$ and $\{A, \mathscr{T}_7\}$ a partition $\{b\} \mid \{a,c\}$. Hence $\{A, \mathscr{T}_4\}$, $\{A, \mathscr{T}_7\}$ are not connected spaces.

Finally, as we saw in SAQ 1, the indiscrete space with underlying set A is connected, whilst the discrete space with underlying set A is not connected.

Thus those topologies $\mathscr{T}$ such that the topological space $\{A, \mathscr{T}\}$ is not connected are the discrete topology, the 3 topologies of the same type as $\mathscr{T}_4$ and the 6 topologies of the same type as $\mathscr{T}_7$. ■

SOLUTION TO SAQ 3

We shall show that

$$T_2 \text{ connected} \implies T_1 \text{ connected}$$

by proving that, if T_1 is not connected, then T_2 is not connected.

So suppose that T_1 is not connected. Then there exists some partition $A \mid B$ of T_1. Since $\mathscr{T}_2$ is finer than $\mathscr{T}_1$, A and B (being open in T_1) are open in T_2; and so $A \mid B$ is a partition of T_2. Therefore T_2 is not a connected space.

On the other hand, it is *not* true in general that

$$T_1 \text{ connected} \implies T_2 \text{ connected}.$$

For let $A = \mathbf{R}$, and take $\mathscr{T}_1$ to be the usual topology, and $\mathscr{T}_2$ to be the discrete topology. Then T_1 is connected, by **S** Theorem 6.2.8, whilst T_2 is not connected, as shown in SAQ 1. ■

SOLUTION TO SAQ 4

By Proposition 7 of *Unit 5*, a subset H of a topological space T is both open and closed in T if and only if $b(H) = \varnothing$. Thus $b(H) \neq \varnothing$ for every non-empty proper subset H of T if and only if there is no non-empty proper subset which is both open and closed in T.

By **S** Corollary 6.2.4, this is the case if and only if T is a connected space. ■

SOLUTION TO SAQ 5 If T is a topological space with the property that every continuous map of T into a discrete space T_1 is constant, then, in particular, there is no continuous map of T *onto* a discrete two-point space; and so T is connected.

Conversely, let T be a connected space, and consider a continuous map $f: T \to T_1$ where T_1 is a discrete space. We must show that f is a constant map. If $c \in f(T)$ then $\{c\}$ is both open and closed in T_1, since T_1 is discrete. Therefore $f^{-1}(c)$ is both open and closed in T, since f is continuous. Since T is a connected space and $f^{-1}(c)$ is non-empty, it follows that

$$f^{-1}(c) = T.$$

Thus $f(x) = c$ for every x in T, i.e. f is a constant map. ■

7.2 Properties of Connected Spaces

READ S page 97, line -14 to page 99, line -4, *omitting* the alternative proof of Proposition 6.2.18 (on page 99).

NOTES

(i) *Page 97, proof of Proposition 6.2.10*
To see that if $A|B$ is a partition of T_2 then $f^{-1}(A)|f^{-1}(B)$ is a partition of T_1, we observe first that $f^{-1}(A)$, $f^{-1}(B)$ are open in T_1, since f is continuous; further, $f^{-1}(A)$, $f^{-1}(B)$ are non-empty, since f is onto. Also

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(T_2) = T_1$$

and

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varnothing) = \varnothing.$$

(ii) *Page 97, Corollary 6.2.12*
The term *graph* was defined on page 28 of *Unit 1*; **S** Proposition 3.6.3 is on page 58.

(iii) *Page 97, line -1*
For 'T' read 'T_1'.

(iv) *Page 98, Corollary 6.2.13*
Combining this result with **S** Proposition 5.5.1, page 83, we see that *if T is a compact connected space and $f: T \to \mathbf{R}$ is continuous then $f(T)$ is a compact interval.*

But compact subsets of $\mathbf{R}$ are closed and bounded. Hence there exist real numbers α, β with $\alpha \leq \beta$ such that $f(T) = [\alpha, \beta]$. In particular, since a bounded closed interval is compact and connected, if f: $[a, b] \to \mathbf{R}$ is continuous then $f([a, b]) = [\alpha, \beta]$ for some α, β.

(v) *Page 98, Proposition 6.2.15*
The proposition does *not* say that the union of any collection $\{A_i\}$ of connected sets is connected; it says that this holds *if* each pair of sets in the collection has a non-empty intersection.

For example, in Example 3(a) we saw that although $(0, 1)$ and $(10, 11)$ are connected, by Theorem 6.2.8, $(0, 1) \cup (10, 11)$ is *not* connected. On the other hand, if $I = \mathbf{N}$, it follows from Proposition 6.2.15 that if *either* $A_{n+1} \subset A_n$ *or* $A_{n+1} \supset A_n$, then $A = \bigcup_{n \in \mathbf{N}} A_n$ is connected.

(vi) *Page 98, Corollary 6.2.16*
In Proposition 6.2.15, we require that each pair of sets in the collection $(A_i)_{i \in I}$ meet, whereas in Corollary 6.2.16 we require only that one set B exists that meets each set in the collection.

(vii) *Page 98, Corollary 6.2.16*
The argument **S** suggests goes as follows. Since B and C_i are connected and $B \cap C_i \neq \varnothing$, it follows from Proposition 6.2.15 that $A_i = B \cup C_i$ is connected for every i in I. Now for all i, j in I, $B \subset A_i \cap A_j$ so that $A_i \cap A_j \neq \varnothing$. Thus

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cup C_i) = B \cup \left(\bigcup_{i \in I} C_i \right)$$

is connected, by Proposition 6.2.15.

(viii) *Page 98, lines -3 and -2*
To see that T_1 is homeomorphic to C_y, consider the bijection $\phi \colon T_1 \to C_y$ given by

$$\phi(x) = (x, y), \qquad x \in T_1.$$

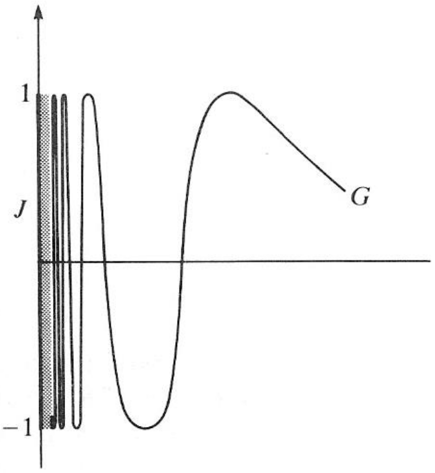
We prove first that ϕ is continuous.

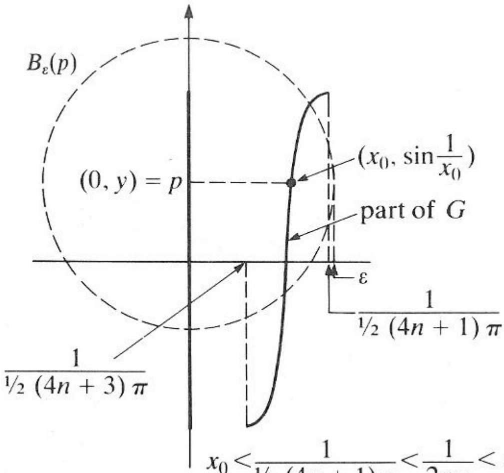
Let q_1, q_2 be the first and second projections

$$q_1 \colon T_1 \times \{y\} \to T_1, \quad q_2 \colon T_1 \times \{y\} \to \{y\}.$$

Since $q_1 \circ \phi$ is the identity function and $q_2 \circ \phi$ is a constant function, both are continuous; hence, by **S** Proposition 3.5.4 (with T', T_2 replaced by T_1, $\{y\}$ respectively), ϕ is continuous. Therefore ϕ is a continuous bijection. Finally, ϕ^{-1} is continuous, since it is the restriction to C_y of the first projection $p_1 \colon T_1 \times T_2 \to T_1$. Hence $\phi \colon T_1 \to C_y$ is a homeomorphism.

(ix) *Page 99, Example 6.2.19*




$B_\varepsilon(p)$
$(0, y) = p$
$(x_0, \sin\frac{1}{x_0})$
part of G
ε
$\frac{1}{\frac{1}{2}(4n+1)\pi}$
$\frac{1}{\frac{1}{2}(4n+3)\pi}$

$$x_0 < \frac{1}{\frac{1}{2}(4n+1)\pi} < \frac{1}{2n\pi} < \varepsilon \quad ▲$$

XAMPLE 5 A map $f \colon S \to S$ is said to have a *fixed point* if there exists c in S such that $f(c) = c$. In this example, we employ the I.V.P. to show that any continuous map $f \colon [a, b] \to [a, b]$ has a fixed point.

We notice first that $f(a) \geq a$ and $f(b) \leq b$. If either $f(a) = a$ or $f(b) = b$ then there is nothing more to do.

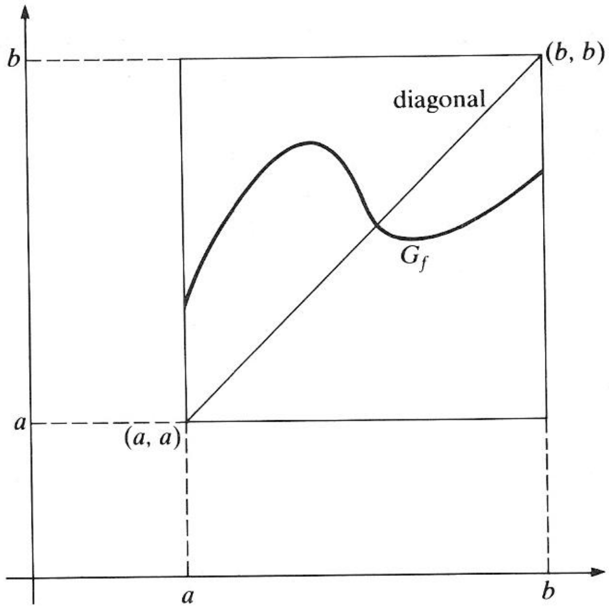
Thus we may suppose that $f(a) > a$ and $f(b) < b$. Now consider the continuous map $g \colon [a, b] \to \mathbf{R}$ given by

$$g(x) = f(x) - x, \qquad x \in [a, b].$$

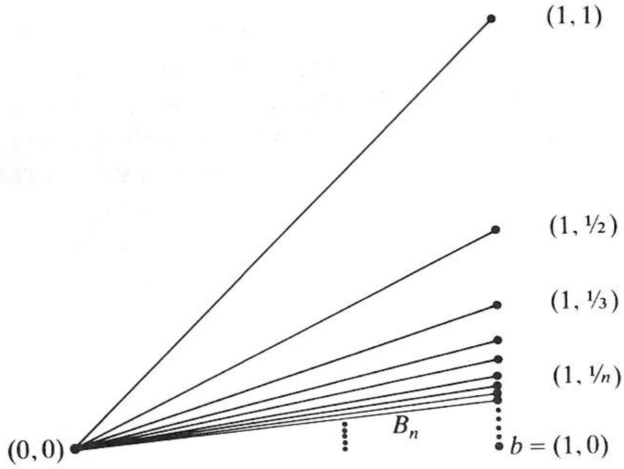
Then $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. Now by **S** Corollary 6.2.14, g has the I.V.P.

Hence there exists c in $[a, b]$ such that $g(c) = 0$. But $g(c) = f(c) - c$, so that $f(c) = c$, i.e. f has a fixed point.

Remark. The result of Example 5 may be interpreted geometrically as saying that the graph of such an f must cross the diagonal of the square $[a,b] \times [a,b]$.



EXAMPLE 6 Consider the following subspace B of $\mathbf{R}^2$: for each positive integer n, let B_n be the line-segment joining the origin to $\left(1, \dfrac{1}{n}\right)$ and let $B_0 = \bigcup_{n=1}^{\infty} B_n$.



Now put

$$B = \{(1,0)\} \cup B_0. \qquad (1)$$

As an illustration of the use of the results of this section (and for applications later in the unit), let us show that B is a connected subspace of $\mathbf{R}^2$ (with the usual topology). (It does not seem intuitively likely, does it?) First notice that B_n is connected for each n, by Corollary 6.2.11, since the line-segment B_n is homeomorphic with a closed interval in $\mathbf{R}$. Since the origin $(0,0)$ is common to all line-segments B_n, it follows from **S** Proposition 6.2.15 that B_0 is connected.

We shall complete the proof by showing that

$$B_0 \subset B \subset Cl(B_0). \qquad (2)$$

It will then follow from **S** Proposition 6.2.18 that B is connected. The left-hand inclusion relation in (2) follows at once from (1); and since $B_0 \subset Cl(B_0)$, it follows that the right-hand inclusion relation in (2) will hold if only we can show that $b = (1,0)$ is a limit point of B_0.

So, let $\varepsilon > 0$ be given, and choose a positive integer m such that $\dfrac{1}{m} < \varepsilon$. Then $\left(1, \dfrac{1}{m}\right) \in B_\varepsilon(b) \cap B_m$. Thus b is a limit point of B_0.

SAQ 6 Prove that if T is a connected space and there exists a non-constant continuous real-valued function on T, then T is uncountable. [Hint: All intervals in $\mathbf{R}$, other than singletons, are uncountable.]

SAQ 7 Let $f\colon \mathbf{R} \to \mathbf{R}$ be a continuous function with the property that

$$(1) \qquad f(x) = f(x + 2\pi) \qquad \text{for each } x \in \mathbf{R}.$$

Prove that for each $a \in \mathbf{R}$ there exists at least one point c in the interval $[a, a + \pi)$ such that

$$(2) \qquad f(c) = f(c + \pi).$$

[Hint: Apply the I.V.P. to the function $g(x) = f(x) - f(x + \pi)$.]

SAQ 8 Prove that if a topological space T contains a connected everywhere-dense subspace, then T is itself connected.

SOLUTION TO SAQ 6 Let $f\colon T \to \mathbf{R}$ be a non-constant continuous function. Since T is connected, by S Corollary 6.2.13, $f(T)$ is an interval. Also, since f is not a constant map, $f(T)$ is not a singleton, and so must be uncountable.

If T were countable, then $f(T)$ would be countable, by Proposition 12 on page 31 of *Unit 1*. Therefore T is uncountable. ■

SOLUTION TO SAQ 7 Let $g(x) = f(x) - f(x + \pi)$, $x \in \mathbf{R}$, and choose any fixed $a \in \mathbf{R}$. Then

$$g(a) = f(a) - f(a + \pi) \qquad \text{(by definition of } g\text{)}$$

and

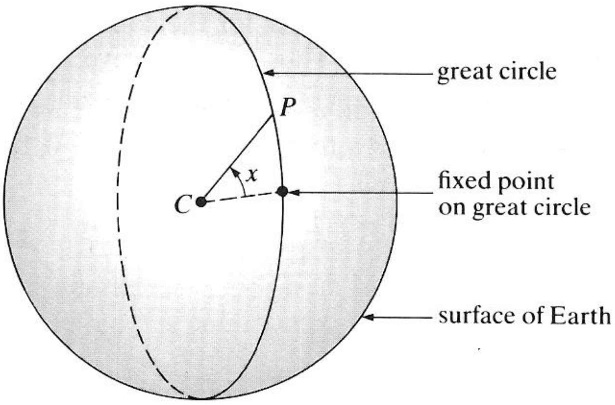
$$\begin{aligned} g(a + \pi) &= f(a + \pi) - f(a + 2\pi) \\ &= f(a + \pi) - f(a) \qquad \text{(by (1));} \end{aligned}$$

so that

$$g(a) = -g(a + \pi).$$

If $g(a) = 0$, we can then choose a c that satisfies (2) to be the point a, as then $f(a) = f(a + \pi)$. On the other hand, if $g(a) \neq 0$, it follows that $g(a)$ and $g(a + \pi)$ are both non-zero and of different signs, so that there must exist some point c between a and $a + \pi$ (that is, with $a < c < a + \pi$) where $g(c)$ vanishes. In other words, $0 = f(c) - f(c + \pi)$, as required. ■

Remark.



By letting x be the angle at the centre of the Earth between a fixed point and a variable point P on a great circle of the Earth, we can interpret this result in the following rather interesting way. Assuming that the temperature $f(x)$ at points on the great circle varies continuously with x, it follows that there are at least two antipodal points on the great circle where the temperatures are the same.

SOLUTION TO SAQ 8 This is an immediate consequence of S Proposition 6.2.18, for if H is an everywhere-dense connected subspace of a topological space T, then $Cl(H) = T$, so that T is connected. ■

7.3 Path-connectedness

READ S from page 99, line -3 to page 103, line 8.

NOTES (i) *Page 100, Definition 6.3.1*
There is a subtle but important point to note in this definition: a path in a topological space T is defined to be a *continuous map* of I into T, and **not** the image of I with respect to a continuous map of I into T.

(ii) *Page 100, line -13*
For 'Exercise 3.9.33(i)' read 'Proposition 1'.

(iii) *Page 101, Figure 6.1*
This figure is drawn in more detail in Note (ix) of the previous section.

(iv) *Page 101, lines -15 to -11*
We observe that if $f_1 = p_1 \circ i \circ f$ and $f_2 = p_2 \circ i \circ f$ then

$$f(t) = (f_1(t), f_2(t)), \qquad t \in I.$$

It follows that if $t \in I$ and $f_1(t) > 0$ then

$$f_2(t) = \sin \frac{1}{f_1(t)}.$$

(v) *Page 102, lines 4 to 12*

At this point you should listen to the tape for Unit 7. ▲

The following result is often used in arguments about path-connectivity.

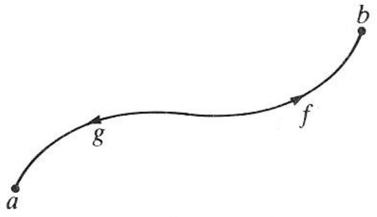
PROPOSITION 2 Let $\equiv$ be the relation defined on a topological space T as follows: if $a, b \in T$ then $a \equiv b$ if there exists a path from a to b in T. Then $\equiv$ is an equivalence relation on T.

PROOF We have to check, in turn, that $\equiv$ has the reflexive, symmetric and transitive properties.

It is easy to see that if $a \in T$ then $a \equiv a$, for the constant path k: $[0,1] \to T$ given by $k(t) = a$, $t \in [0,1]$, joins a to a. Thus $\equiv$ is reflexive.

Next we show that, if $a, b \in T$ and $a \equiv b$, then $b \equiv a$. This is achieved by 'reversing' the path which joins a to b. In detail, let f: $[0,1] \to T$ be a continuous function such that $f(0) = a$, $f(1) = b$, and define g: $[0,1] \to T$ by putting

$$g(t) = f(1-t), \qquad t \in [0,1].$$



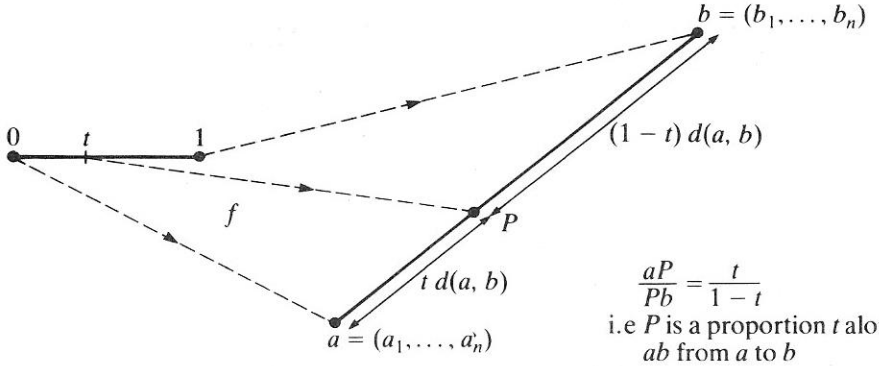
Then g is continuous, by **S** Proposition 3.1.8, page 48, since it is the composite of f and the continuous map $t \mapsto 1-t$, $t \in [0,1]$, of $[0,1]$ to itself. Since $g(0) = f(1) = b$ and $g(1) = f(0) = a$, g is a path from b to a in T. Thus $b \equiv a$.

Finally, it follows from **S** Lemma 6.3.3 that, if $a, b, c \in T$ and $a \equiv b$, $b \equiv c$, then $a \equiv c$; so that $\equiv$ is transitive. Hence $\equiv$ is an equivalence relation. ■

Results about path-connectivity of subsets of Euclidean space are often proved by using line-segments joining points. If $a, b \in \mathbf{R}^n$, then the line-segment joining a and b is the set of points P of $\mathbf{R}^n$ of the form $(1-t)a + tb$, where $0 \le t \le 1$.

Note that, if $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, then

$$(1 - t)a + tb = ((1 - t)a_1 + tb_1, \ldots, (1 - t)a_n + tb_n).$$



$\frac{aP}{Pb} = \frac{t}{1 - t}$
i.e P is a proportion t along ab from a to b

If we define $f\colon [0, 1] \to \mathbf{R}^n$ by

$$f(t) = (1 - t)a + tb, \qquad 0 \le t \le 1,$$

then f is a map such that $f(0) = a$ and $f(1) = b$. To show that the line-segment joining a and b is indeed a path from a to b in $\mathbf{R}^n$ we require:

POSITION 3 If $a, b \in \mathbf{R}^n$ then

$$f\colon [0, 1] \to \mathbf{R}^n$$

given by

$$f(t) = (1 - t)a + tb, \qquad t \in [0, 1],$$

is continuous (where $[0, 1]$, $\mathbf{R}^n$ have the Euclidean metrics).

PROOF We can assume that $a \ne b$, for if $a = b$ then f is a constant map and there is nothing to prove.

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$; and for t in $[0, 1]$ let

$$f(t) = (f_1(t), \ldots, f_n(t)).$$

Then

$$f_i(t) = (1 - t)a_i + tb_i, \quad i = 1, \ldots, n.$$

Thus, if $s, t \in [0, 1]$, then

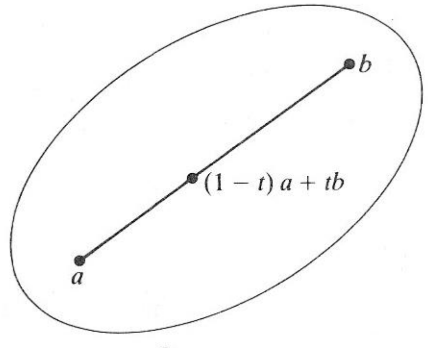
$$f_i(s) - f_i(t) = (t - s)(a_i - b_i), \quad i = 1, \ldots, n.$$

Now, if d denotes the Euclidean metric on $\mathbf{R}^n$ then

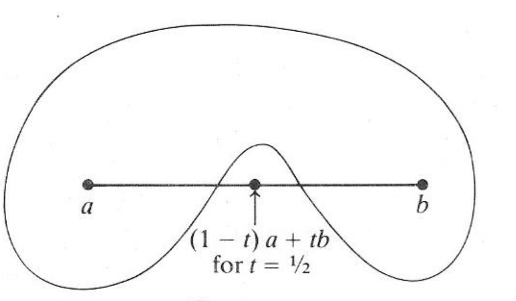
$$\begin{aligned} d(f(s), f(t)) &= \left(\sum_{i=1}^{n} (f_i(s) - f_i(t))^2 \right)^{1/2} \\ &= |t - s| \left(\sum_{i=1}^{n} (a_i - b_i)^2 \right)^{1/2} \\ &= |s - t| d(a, b). \end{aligned}$$

Put $d(a, b) = K > 0$, and let $\varepsilon > 0$ be given. If $s, t \in [0, 1]$ and $|s - t| < \varepsilon/K$, then $d(f(s), f(t)) < \varepsilon$. Therefore f is continuous. ■

XAMPLE 7 A subset C of $\mathbf{R}^n$ is said to be *convex* if whenever $a, b \in C$ then $(1 - t)a + tb \in C$ for all real numbers t such that $0 \le t \le 1$; i.e. C is a convex subset of $\mathbf{R}^n$ if any two points in C can be joined by a line-segment in C.



C convex

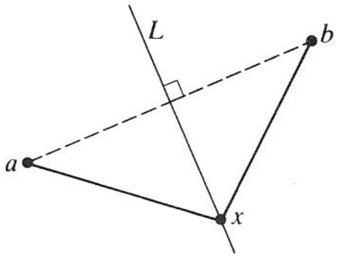


C not convex

It follows from Proposition 3 that every convex subset of $\mathbf{R}^n$ is path-connected. In particular, $\mathbf{R}^n$ is path-connected for every $n \geq 1$.

EXAMPLE 8 If K is a countable subset of $\mathbf{R}^2$ then the subspace $\mathbf{R}^2 - K$ is path-connected.

Put $T = \mathbf{R}^2 - K$. We must show that, if a, b are distinct points of T, then there is a path from a to b in T.



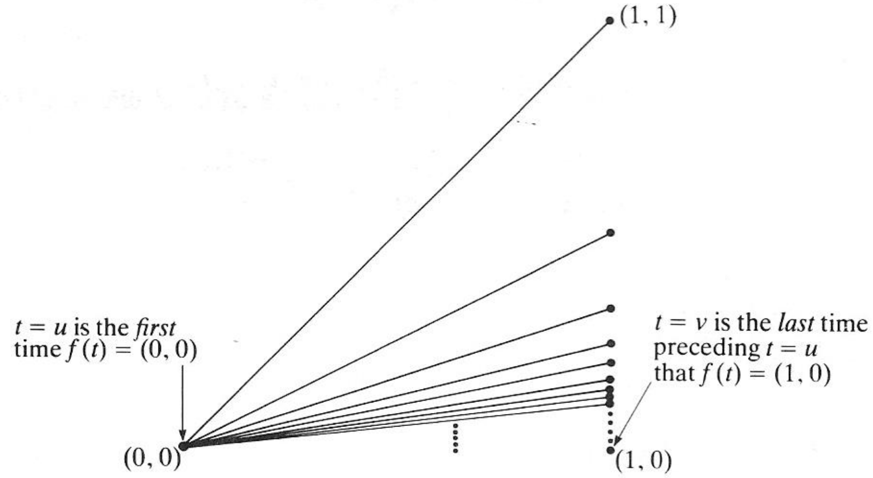
Let L be the perpendicular bisector of the line-segment joining a to b, and, for each x in L, let L_x be the subset of $\mathbf{R}^2$ which is the union of the line-segment joining a to x and the line-segment joining x to b. Since $\{x : x \in L\}$ is uncountable, there are uncountably many such subsets L_x of $\mathbf{R}^2$. Also, if $x, y \in L$ and $x \neq y$, then $L_x \cap L_y = \{a, b\}$. Since K is countable, we see that there is some x in L such that $L_x \cap K = \varnothing$ and so $L_x \subset T$.

By applying **S** Proposition 6.3.3, we see that L_x is the image of a path from a to b in T. Therefore T is path-connected.

Remark. A similar argument shows that, if $n > 1$ and K is a countable subset of $\mathbf{R}^n$, then the subspace $\mathbf{R}^n - K$ is path-connected.

EXAMPLE 9 The connected subspace B of $\mathbf{R}^2$, introduced in Example 6, is not path-connected.

To prove this, we shall show that the assumption that there is a path joining $(1, 0)$ and $(0, 0)$ in B leads to a contradiction.


(1, 1)
$t = u$ is the *first* time $f(t) = (0, 0)$
$t = v$ is the *last* time preceding $t = u$ that $f(t) = (1, 0)$
(0, 0)
(1, 0)

For $v < t < u$, $f(t)$ is 'somewhere between' $(1, 0)$ and $(0, 0)$.

Suppose that $f\colon [0, 1] \to B$ is a continuous map such that $f(0) = (1, 0)$ and $f(1) = (0, 0)$. Put

$$f(t) = (f_1(t), f_2(t)), \qquad t \in [0, 1].$$

The subset

$$C_0 = \{t \in [0, 1] : f_1(t) = 0\}$$

of $[0, 1]$ is non-empty, since $1 \in C_0$; and it is closed. Thus if $u = \inf C_0$, then $u \in C_0$ (by **S** Example 3.7.13), so that $f_1(u) = 0$. We note that $u > 0$ since $f_1(0) = 1$.

Next, let

$$C_1 = \{t \in [0, u] : f_1(t) = 1\}.$$

Then C_1 is non-empty, since $0 \in C_1$, and it is closed; so that if $v = \sup C_1$ then $f_1(v) = 1$. Since $0 \leq v \leq u$ and $f_1(v) = 1 \neq 0 = f_1(u)$, we see that $0 \leq v < u$.

Now choose a real number w such that $v < w < u$. If $t \in [0, w]$, then $f_1(t) > 0$, by definition of u, so that the real-valued function g on $[0, w]$, given by

$$g(t) = \frac{f_2(t)}{f_1(t)}, \qquad t \in [0, w],$$

is well-defined and continuous. Now if $t \in [0, w]$ then either $f(t) = (1, 0)$, in which case $g(t) = 0$, or $f(t) \in B_n$ for some positive integer n, in which case $g(t) = \frac{1}{n}$. Thus

$$g \colon [0, w] \to \mathbf{R}$$

is a continuous function which can take only the values $0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$. Since $g(0) = 0$ we must have $g(t) = 0$ for all t in $[0, w]$, for otherwise the I.V.P. would not hold for g on $[0, w]$.

But $g(t) = 0$ if and only if $f(t) = (1, 0)$ so that we see that $f_1(t) = 1$ for all t in $[0, w]$. In particular, $f_1(w) = 1$. Since $v < w < u$, this is a contradiction of the definition of v.

Therefore no such path f in B exists.

SAQ 9 Let

$$\begin{aligned} H_1 &= \{(x_1, x_2) \in \mathbf{R}^2 : x_1 x_2 > 1\}, \\ H_2 &= H_1 \cup \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = x_2\}, \text{ and} \\ H_3 &= H_2 - \{(0, 0)\}. \end{aligned}$$

Which of the subspaces H_1, H_2, H_3 of $\mathbf{R}^2$, with the usual topology, are (i) connected; (ii) path-connected?

SAQ 10 Which of the following subspaces of $\mathbf{R}^2$ (with the usual topology) are (i) connected; (ii) path-connected?

(a) K_1, consisting of all points with at least one irrational coordinate

(b) K_2, consisting of all points with at least one rational coordinate

(c) $K_3 = \mathbf{Q} \times \mathbf{Q}$

SAQ 11 (a) Prove that if $f \colon T_1 \to T_2$ is continuous and T_1 is a path-connected space, then $f(T_1)$ is path-connected.

(b) Prove that if $(C_i)_{i \in I}$ is a family of path-connected subspaces of a topological space T, and B is a path-connected subspace of T such that $B \cap C_i \neq \varnothing$ for every i in I, then $B \cup \bigcup_I C_i$ is path-connected.

(c) Prove that if T_1 and T_2 are path-connected spaces, then the topological product $T_1 \times T_2$ is path-connected.

***SAQ 12** Show that any two points in a connected open subset of $\mathbf{R}^n$ can be joined by a polygonal path (that is, by a finite union of line-segments end to end).

[Hint: Use the same method as that in the proof of **S** Proposition 6.4.2.]

SOLUTION TO SAQ 9 A sketch of H_1 suggests that it is probably neither connected nor path-connected. There are various ways of *proving* this; possibly the simplest is by use of Corollary 6.2.4, **S** page 95.

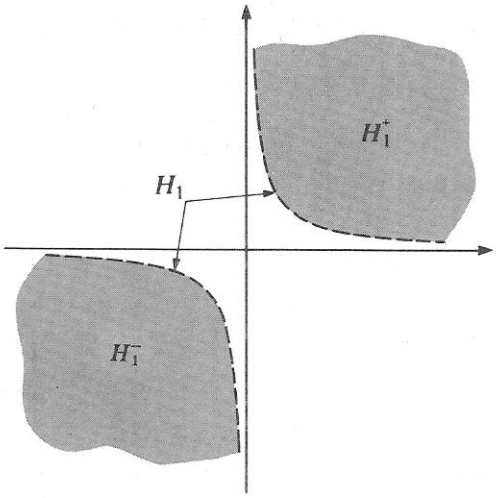
Let

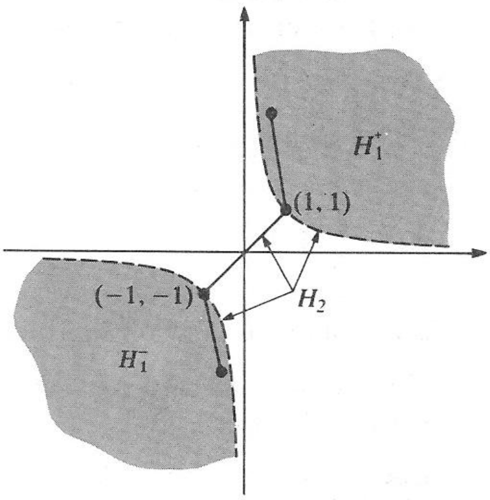
$$H_1^+ = \{(x_1, x_2) \in H_1 : x_1 > 0\}.$$

Since

$$\begin{aligned} H_1^+ &= H_1 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_1 > 0\} \\ &= H_1 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0\}, \end{aligned}$$

we see that H_1^+ is a non-empty proper subset which is both open and closed in H_1. Therefore H_1 is not connected, and hence (by Proposition 6.4.1) is not path-connected.





Now let us deal with H_2, which *looks* connected and path-connected. Both H_1^+ and

$$H_1^- = \{(x_1, x_2) \in H_1 : x_1 < 0\}$$

are convex subsets of $\mathbf{R}^2$ and so are path-connected. But each point of H_1^+ can be joined to $(1, 1)$ by a line-segment in H_2, and each point of H_1^- can be joined to $(-1, -1)$ by a line-segment in H_2. It follows (by the transitive property of path-connectedness) that H_2 is path-connected.

Hence, by Proposition 6.4.1, H_2 is also connected.

Finally, let us consider H_3. Put

$$H_3^+ = \{(x_1, x_2) \in H_3 : x_1 > 0\}.$$

Then

$$\begin{aligned} H_3^+ &= H_3 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_1 > 0\} \\ &= H_3 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0\}. \end{aligned}$$

Thus H_3^+ is a non-empty proper subset which is both open and closed in H_3. Therefore H_3 is not connected, and so is not path-connected. ■

SOLUTION TO SAQ 10

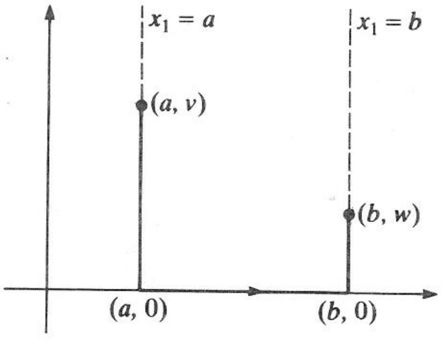
(a) $K_1 = \mathbf{R}^2 - (\mathbf{Q} \times \mathbf{Q})$; so that, since $\mathbf{Q} \times \mathbf{Q}$ is a countable set (by SAQ 26 on page 32 of *Unit 1*), it follows from Example 8 that K_1 is path-connected and so connected.

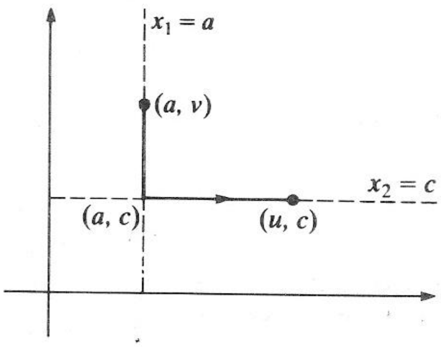
(b) Let (a, v) and (b, w) be points of K_2 where a, b are rational. The line-segment joining (a, v) to $(a, 0)$ lies in K_2 (since the first coordinate of each point of the line-segment is equal to a). Also, the line-segments joining $(a, 0)$ to $(b, 0)$ and joining $(b, 0)$ to (b, v) lie in K_2. Thus there is a path in K_2 joining (a, v) to (b, w).

A similar argument shows that two points which both have rational second coordinate can be joined by a path in K_2.

Finally, let (a, v) and (u, c) be points of K_2 where a and c are rational. Then the line-segment joining (a, v) to (a, c) and the line-segment joining (a, c) to (u, c) both lie in K_2. Hence there is a path from (a, v) to (u, c) in K_2.

Therefore K_2 is path-connected and so, by Proposition 6.4.1, is connected.



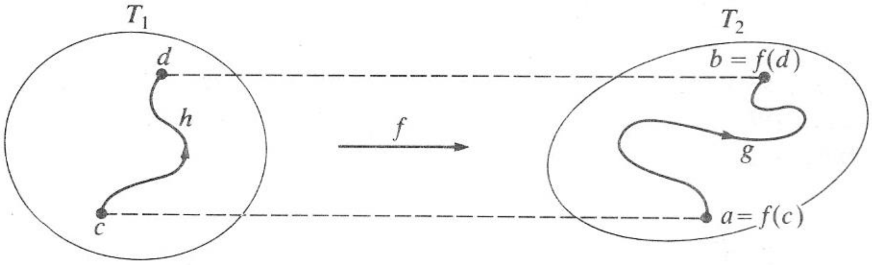


(c) If $\mathbf{Q} \times \mathbf{Q}$ were connected, then it would follow from **S** Proposition 6.2.17 that $\mathbf{Q}$ is connected. Since $\mathbf{Q}$ is not connected (Example 3(b)), we see that $\mathbf{Q} \times \mathbf{Q}$ is not connected, and so is not path-connected. ■

SOLUTION TO SAQ 11

(a) As in the proof of Proposition 6.2.10, we may assume that f is a continuous map onto T_2, and then prove that T_2 is path-connected.

Let a and b be points of T_2. We must find a path from a to b in T_2.



Since f is onto, there exist c, d in T_1 such that $f(c) = a$ and $f(d) = b$. Since T_1 is path-connected, there exists a continuous map h: $[0,1] \to T_1$ such that $h(0) = c$ and $h(1) = d$. Now put

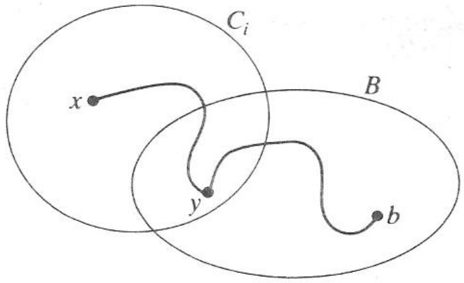
$$g = f \circ h\colon [0,1] \to T_2.$$

Then g is continuous and

$$g(0) = f(h(0)) = f(c) = a, \quad \text{and} \quad g(1) = f(h(1)) = f(d) = b.$$

Thus g is a path from a to b in T_2.

(b) Put $D = B \cup \bigcup_{i \in I} C_i$ and take a particular point b of B. To prove that D is path-connected it is enough, in view of Proposition 2 above, to show that each point of D can be joined to b by a path in D.



Plainly, each point of B can be joined to b by a path in D.

Now suppose that $x \in C_i$ and choose a point y of $B \cap C_i$. There is a path in C_i, and hence in D, joining x and y; and there is a path in B, and hence in D, joining y and b. So, by **S** Lemma 6.3.3, there is a path in D joining x and b. Therefore D is path-connected.

(c) Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be points of $T_1 \times T_2$. We must find a path from a to b in $T_1 \times T_2$.

Since T_1 is path-connected, there exists a path from a_1 to b_1 in T_1; that is, there exists a continuous function f_1: $[0,1] \to T_1$ such that $f_1(0) = a_1$ and $f_1(1) = b_1$. Similarly, since T_2 is path-connected, there exists a continuous function f_2: $[0,1] \to T_2$ such that $f_2(0) = a_2$ and $f_2(1) = b_2$.

By **S** Proposition 3.5.4, the function

$$f\colon [0,1] \to T_1 \times T_2$$

given by $f(t) = (f_1(t), f_2(t))$, $t \in [0,1]$, is continuous.

Also

$$f(0) = (f_1(0), f_2(0)) = (a_1, a_2) = a$$

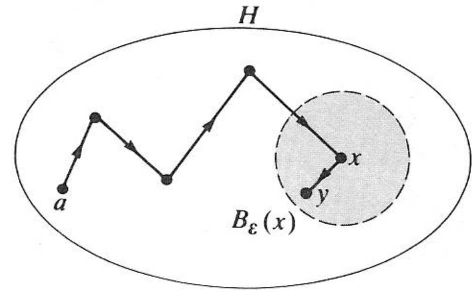
and

$$f(1) = (f_1(1), f_2(1)) = (b_1, b_2) = b.$$

Thus f is a path from a to b in $T_1 \times T_2$. Hence $T_1 \times T_2$ is path-connected. ■

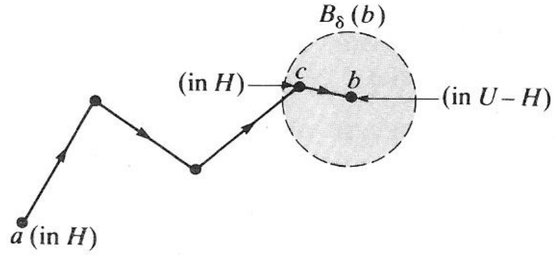
SOLUTION TO SAQ 12 Let U be a connected open subset of $\mathbf{R}^n$ and let a be a particular point of U. It suffices to show that each point of U can be joined to a by a polygonal path.

Let H be the set of points of U which can be joined to a by a polygonal path in U. By **S** Corollary 6.2.4, we may prove that $H = U$ by showing that the non-empty subset H of U is both open and closed in U.



If $x \in H$ then $x \in U$ so that, since U is open in $\mathbf{R}^n$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. Each point y of $B_\varepsilon(x)$ can be joined to x by a line-segment. But x can be joined to a by a polygonal path in U, so that each point of $B_\varepsilon(x)$ can be joined to a by a polygonal path in U. Hence $B_\varepsilon(x) \subset H$. Therefore H is open in $\mathbf{R}^n$, and so open in U.

Now let b be a point of $U - H$. Since U is open in $\mathbf{R}^n$ and $b \in U$, there exists $\delta > 0$ such that $B_\delta(b) \subset U$.



If $B_\delta(b) \cap H \neq \varnothing$ and $c \in B_\delta(b) \cap H$, then there exists a polygonal path in U joining a to c. But the line-segment joining c to b lies in $B_\delta(b)$, and so we obtain a polygonal path in U joining a to b, which is a contradiction since (by hypothesis) $b \notin H$. Thus $B_\delta(b) \subset U - H$. Hence $U - H$ is open in $\mathbf{R}^n$, and so open in U.

Therefore H is a non-empty subset of the connected space U and H is both open and closed in U. Hence $H = U$, as required. ■

7.4 Components

In this section, we find that every topological space T can be expressed as the union of disjoint connected subspaces; we call these the *components* of T. The components of T are the maximal connected subspaces of T, in the sense that every connected subspace is contained in a component of T.

READ S from the beginning of Section 6.5 on page 103 to the bottom of page 104, omitting the passage on page 104 marked †.

NOTES

(i) *Page 103, lines 10–14*
Since the components of T are equivalence classes with respect to the relation $\sim$ on T, they are mutually disjoint and their union is equal to T. If C_x and C_y are the components containing points x and y of T, then either $C_x = C_y$ or C_x and C_y are disjoint.

(ii) *Page 103, lines 16 and 17*
Ignore the sentence in parentheses and, instead, read the following *additional text.*

By saying that C is a maximal connected subspace of T, we mean that C is connected and that there is no connected subspace of T which contains C as a proper subset. Let us show that the components of T are precisely its maximal connected subspaces.

Firstly, observe that if D is a connected subspace of T then D is contained in some component of T. To see this, take a particular point x of D. If $y \in D$, then x and y belong to the connected subspace D so that $x \sim y$ and hence y belongs to the component C_x of T which contains x. Therefore $D \subset C_x$.

Now let D be a maximal connected subspace of T. Since D is connected, it is contained in some component C of T, and since D is maximal, it follows that $D = C$. Thus we have shown that a maximal connected subspace is necessarily a component of T.

Conversely, let C be a component of T and let D be a connected subspace of T such that $C \subset D$. Since D is connected, there exists a component C' of T such that $D \subset C'$. Since components of T are either the same or are disjoint, we see that $C = C'$. But $C \subset D \subset C'$ and so $C = D$. Therefore C is a maximal connected subspace of T.

(iii) *Page 103, lines −8 to −1*
To define a set map

$$\mathscr{H}(f): \mathscr{H}(T_1) \to \mathscr{H}(T_2),$$

we must associate a component $\mathscr{H}(f)(C)$ of T_2 with each component C of T_1. An alternative description of the map $\mathscr{H}(f)$, which makes it clear that it is well-defined, goes as follows.

Let C be a component of T_1. Then C is connected, so that $f(C)$ is a connected subset of T_2 and hence is contained in a component of T_2. We define $\mathscr{H}(f)(C)$ to be the component of T_2 which contains $f(C)$.

This agrees with S's definition; for, if $x \in T_1$ and C_x is the component of T_1 which contains x, then $f(x) \in f(C_x)$ so that $f(C_x)$ is a connected set containing $f(x)$. Hence the component $C_{f(x)}$ of T_2 which contains $f(x)$ also contains $f(C_x)$. Therefore

$$\mathscr{H}(f)(C_x) = C_{f(x)}.$$

(iv) *Page 103, line −1*
In the 1975 edition of S, this should read '$C_{f(x)} = C_{f(y)}$'.

(v) *Page 104, Proposition 6.5.2*
The statement that the 'displayed diagram commutes' means simply that

$$\mathscr{H}(g) \circ \mathscr{H}(f) = \mathscr{H}(g \circ f).$$

Thus it must be shown that for each component C of T_1 we have

$$(\mathscr{H}(g) \circ \mathscr{H}(f))(C) = \mathscr{H}(g \circ f)(C).$$

(vi) *Page 104, line −13, Proof of Theorem 6.5.4*
If $f: T_1 \to T_2$ is a homeomorphism, then f is a one–one correspondence such that both f and f^{-1} are continuous. From the discussion in Section 1.3 of *Unit 1*, we know that

$$f \circ f^{-1} = 1_{T_2} \quad \text{and} \quad f^{-1} \circ f = 1_{T_1},$$

where 1_{T_1}, 1_{T_2} denote the identity maps of T_1, T_2 respectively. By S Propositions 6.5.2 and 6.5.3,

$$\mathscr{H}(f) \circ \mathscr{H}(f^{-1}) = \mathscr{H}(f \circ f^{-1}) = \mathscr{H}(1_{T_2}) = 1,$$

where 1 denotes the identity map of the set $\mathscr{H}(T_2)$. Similarly, $\mathscr{H}(f^{-1}) \circ \mathscr{H}(f) = 1$, the identity map of $\mathscr{H}(T_1)$. From the Remark on page 22 of Section 1.3 of *Unit 1*, it follows that $\mathscr{H}(f)$ is a one–one correspondence and $\mathscr{H}(f^{-1})$ is the inverse of $\mathscr{H}(f)$.

(vii) *Page 104, line −7*
We ask you to complete the analysis of **S** Example 3.6.2(d) below, in SAQ 16.

(viii) *Page 104, lines −6 to −4*
The topological invariance of the number of components is often exploited in the following way.

Suppose that T_1 and T_2 are topological spaces, and that there exists some point a of T_1 such that, for every point b of T_2, the subspaces $T_1 - \{a\}$ and $T_2 - \{b\}$ have different numbers of components; then T_1 and T_2 are *not homeomorphic.*

For, if there exists a homeomorphism $h: T_1 \rightarrow T_2$, then (by SAQ 25 of *Unit 5*) the subspaces $T_1 - \{a\}$ and $T_2 - \{f(a)\}$ are homeomorphic and so have the same number of components, contrary to the hypothesis. This technique is used below in Example 10 and in SAQ 16.

(ix) *Page 104, lines −3 to −1*
If C is a component of a topological space T, then C is connected; so that, by **S** Proposition 6.2.18, $Cl(C)$ is connected. Since $C \subset Cl(C)$ and C is a maximal connected subspace, it follows that $C = Cl(C)$ and hence C is closed in T.

Thus the components of a topological space T form a disjoint covering of T by closed sets. ▲

In particular spaces T, the definition of component given by **S** on page 103 is sometimes awkward to use. The following criterion is sometimes helpful.

PROPOSITION 4 A subset H of a topological space T is a component of T if it satisfies the following two conditions:

(a) H is both open and closed in T; and

(b) H is connected.

PROOF Since H is connected, all we need verify is that H is maximal; that is, that there exists no connected subset, C say, of T such that H is a proper subset of C.

So suppose that such a subset C exists. Since H is both open and closed in T, it is also both open and closed in C. Since C is connected, this is a contradiction. Thus H must be maximal and so a component of T.

Remark. Conditions (a) and (b) of Proposition 4 together imply that H is a component of T. That is, Proposition 4 supplies *sufficient* conditions for a subset H of a topological space T to be a component of T.

It is most important to note, however, that the conditions are *not necessary* for H to be a component of T. Of course, if H is a component of T, then H is connected (by definition) and closed in T (by Note (ix) above); but it is not necessarily the case that H is open in T.

For example, in **S** Example 6.5.1 we saw that the components of $\mathbf{Q}$, with the usual topology, are the singleton subsets of $\mathbf{Q}$; and no singleton is open in $\mathbf{Q}$. You will find another counter-example in SAQ 14 below.

EXAMPLE 10 No two of the three intervals $[-1, 1]$, $(-1, 1]$ and $(-1, 1)$ are homeomorphic.

Firstly, $[-1, 1]$ is not homeomorphic to either of the other two intervals, since $[-1, 1]$ is compact whilst neither of the others is compact.

Next, $(-1, 1] - \{1\}$ is connected; whilst if b is any point of $(-1, 1)$, then $(-1, 1) - \{b\}$ has two components. By the discussion in Note (viii) above, we see that $(-1, 1]$ and $(-1, 1)$ are not homeomorphic.

Remark. Each non-singleton interval in **R** is homeomorphic to precisely one of the 'standard' intervals considered in Example 10. Thus these standard intervals are representatives of the three homeomorphism classes of intervals in **R**.

SAQ 13 Let $T = \{A, \mathscr{T}\}$ where $A = \{a, b, c, d, e\}$ and

$$\mathscr{T} = \{\varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, A\}.$$

Find the components of T.

SAQ 14 Let $S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and let T be the subspace $S \times [0, 1]$ of $\mathbf{R}^2$ (with the usual topology). Find the components of T. [Hint: You may find Proposition 4 useful.]

Which components of T are open as well as closed?

SAQ 15 Prove that **R** and $\mathbf{R}^2$ (with the usual topologies) are not homeomorphic.

SAQ 16 Complete the analysis of **S** Example 3.6.2(d), page 57.

***SAQ 17** (a) Prove that if a topological space T has a finite number of components, then each component is both open and closed in T.

(b) Prove that if all the components of a compact space T are open in T, then there are only finitely many of them.

SOLUTION TO SAQ 13 Firstly, we observe that $\{a\} \mid \{b, c, d, e\}$ is a partition of T; and so T is not connected.

Now put $H = \{b, c, d, e\}$. By Definition 3.4.1, **S** page 51, the induced topology on H as a subspace of T is

$$\{\varnothing, \{c, d\}, H\}.$$

The only non-empty proper closed subset of H is $\{b, e\}$. Thus H has no non-empty proper subsets which are both open and closed in H. Therefore H is connected.

Since the only subset of T that properly contains H is T itself, and T is not connected, it follows from the definition on **S** page 103 of a *component* as a maximal connected subset that H is a component of T.

Since $T - H = \{a\}$, which is connected (for example, in view of Definition 6.2.2), $\{a\}$ is the other component of T. ■

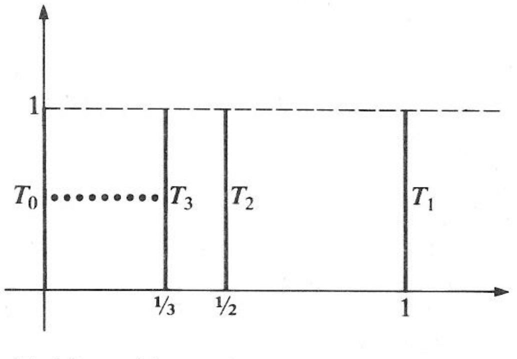
SOLUTION TO SAQ 14 Let n be a positive integer and put $T_n = \left\{\frac{1}{n}\right\} \times [0, 1]$. Since T_n is closed in $\mathbf{R}^2$, it is closed in T (by Proposition 3.7.6, **S** page 60).

But

$$T_n = T \cap \left(\left(\frac{1}{n+1}, \frac{1}{n-1}\right) \times \mathbf{R}\right),$$

and so T_n is open in T. Therefore T_n is both open and closed in T.

Also, T_n is connected since it is the product of two connected sets (by Proposition 6.2.17, **S** page 98). It follows from Proposition 4 that each T_n is a component of T.



Now let $T_0 = \{0\} \times [0, 1]$, and let C be the component of T that contains T_0.

Since none of the components T_n meet T_0, the component C that contains T_0 must be disjoint from all the T_n (by Note (i) on page 70). Hence, since

$$T = T_0 \cup \left(\bigcup_{n=1}^{\infty} T_n \right),$$

it follows that $C \subset T_0$; and so, since also $T_0 \subset C$, we see that $C = T_0$. In other words, T_0 is also a component of T. Thus the components of T are T_0 and the subsets $T_n, n \in \mathbf{N}$. Every component of T except T_0 is open and closed in T. ■

SOLUTION TO SAQ 15 The subspace $\mathbf{R} - \{0\}$ of $\mathbf{R}$ has two components, namely R_+ and R_-. But, by the result of Example 8, for b in $\mathbf{R}^2$ the subspace $\mathbf{R}^2 - \{b\}$ of $\mathbf{R}^2$ is path-connected and so connected. By the method of Note (viii) we see that $\mathbf{R}$ and $\mathbf{R}^2$ are not homeomorphic. ■

SOLUTION TO SAQ 16 The argument suggested by **S** on page 57 can now be carried through as follows. There exists a in T such that $T - \{a\}$ has 3 components. But for every b in L, either $L - \{b\}$ is connected or $L - \{b\}$ has 2 components. Hence T and L are not homeomorphic. ■

SOLUTION TO SAQ 17

(a) Let $C_1, \ldots, C_n$ be the components of T. For each i, C_i is a closed set of T (**S** page 104, line -3). It follows that $\bigcup_{j \neq i} C_j$ is a closed set of T. But

$$C_i = T - \bigcup_{j \neq i} C_j,$$

and so C_i is open in T. Therefore each component C_i of T is both open and closed in T.

(b) The components of T form a disjoint covering $\mathscr{C}$ of T. If each component of T is open, then $\mathscr{C}$ is an open covering of T. Also, no proper subset of $\mathscr{C}$ is a covering of T.

Suppose that T possesses infinitely many components. Then the covering $\mathscr{C}$ contains infinitely many open sets; and since T is compact, there exists a finite subcover of T. This contradicts the result of the previous paragraph. It follows that T possesses only finitely many components. ■

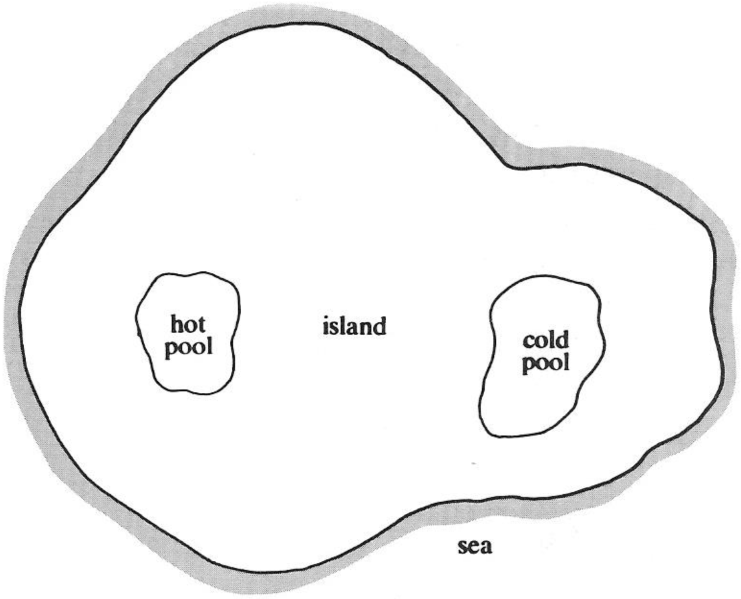
7.5 Epilogue (non-assessed)

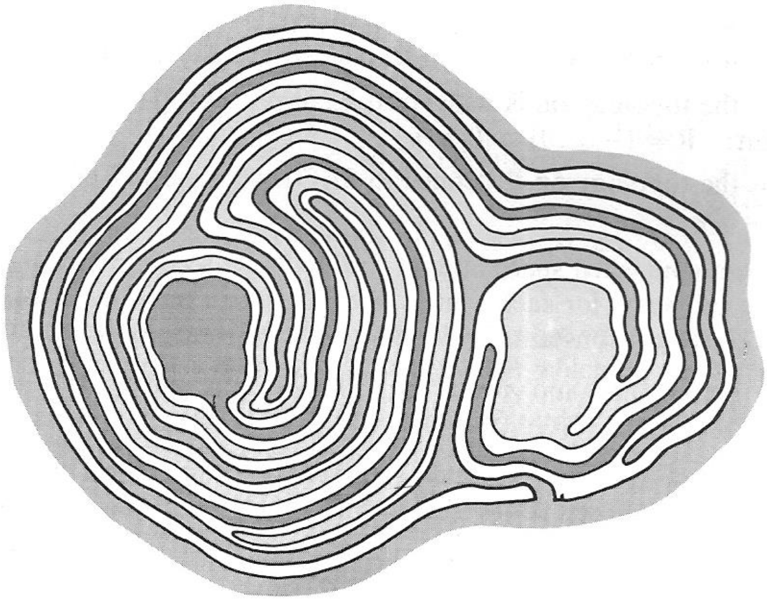
Let C be a subset of $\mathbf{R}^2$ that is homeomorphic with the circle $\{(x_1, x_2): x_1^2 + x_2^2 = 1\}$. It is a deep theorem in topology, called the *Jordan Curve Theorem*, that $\mathbf{R}^2 - C$ then consists of two path-connected open sets, 'an inside' and 'an outside', both of which have C as their boundary.

The following example, known as the *Lakes of Wada*, shows that $\mathbf{R}^2$ can be more devious than this theorem suggests! We construct three non-empty disjoint path-connected open sets in $\mathbf{R}^2$, all of which have the same boundary C, and which, together with C, fill all of $\mathbf{R}^2$. For clarity we explain the construction in a rather picturesque way.

EXAMPLE 11 An Open University student (having emigrated) now lives on an island in the southern Pacific. On his island are two fresh-water pools; one of them contains hot water (as the island is of volcanic origin), and the other cold water. The student would like to bring each source of water to a short distance from each point on the island. And why not?!

He starts by digging ditches that do not intersect; first a ditch from the sea that winds about so that it comes within one metre of each point of (what's left of) the island, then a ditch from the hot pool with the same property, and then a ditch from the cold pool. Then every point of the island is within 1 m of all three types of water supply.





Being a perfectionist (like all OU students), he is then not satisfied, as he has noticed that he can do even better. He extends each ditch in turn in the same way, so that the triple water supply comes within 10^{-1} m of each remaining point on the island, repeats the process so that it comes within 10^{-2} m of each remaining point, and so on. He continues this process indefinitely (life expectancy on this island is infinity!).

Let the set S_1 consist of the sea plus the final ditch containing sea-water, and S_2 and S_3 the hot and cold pools plus their respective final ditches.

As every point on the island is now a zero distance from the triple water supply, it is easy to convince yourself that

(i) S_1, S_2 and S_3 are disjoint path-connected open sets;

(ii) S_1, S_2 and S_3 all have the same boundary, C say (recall the comment after the Solution to SAQ 16 in Section 5.3 of *Unit* 5);

(iii) $\mathbf{R}^2 = S_1 \cup S_2 \cup S_3 \cup C$ (in other words, all that is left of the original island is C — and C contains no ball at all, since each point of C is a zero distance from each of S_1, S_2 and S_3).

7.6 Objectives of Unit 7

After studying the unit, you should

(1) know the three characterizations of connectedness;

(2) understand the proof that a connected subset of $\mathbf{R}$ (with the usual topology) must be an interval;

(3) know the basic properties of connected spaces and that connectedness is a topological property;

(4) know the definition of path-connectedness, and understand the relationship between connectedness and path-connectedness, in particular understand why 'the topologists' sine-curve' is connected but not path-connected;

(5) know the two definitions of component, and be able to determine the components of a space;

(6) know that the number of components is a topological property.

7.7 Revision Questions

SAQ 18 In which of the following cases is $T = \{\mathbf{R}, \mathscr{T}\}$ connected?

(a) $\mathscr{T}$ is the 'left-ray topology' on $\mathbf{R}$, defined in SAQ 22 on page 112 of *Unit 4*.

(b) $\mathscr{T}$ is the topology on $\mathbf{R}$ defined in Example 9 on page 104 of *Unit 4*. [Hint: $\mathbf{R} = (-\infty, 0) \cup [0, \infty)$.]

(c) $\mathscr{T}$ is the topology on $\mathbf{R}$ defined in SAQ 6 on page 101 of *Unit 4*, with $\mathscr{T} = \mathscr{T}_0$.

SAQ 19 Let T_1 be a connected space and let $f: T_1 \to T_2$ be a continuous map with the following property: for each x in T_1 there exists an open set V_x of T_1 such that $x \in V_x$ and $f \mid V_x$ is constant. Prove that f is a constant map.

[Hint: Let $c \in f(T_1)$ and $A = f^{-1}(c)$. Show that A and $T_1 - A$ are both open in T_1.]

SAQ 20 (a) Prove that, if C is a connected subspace of a topological space T and H is an open and closed set of T such that $C \cap H \neq \varnothing$, then $C \subset H$.

(b) Prove that, if $(C_n)_{n \in \mathbf{N}}$ is a sequence of (non-empty) connected subspaces of a topological space such that $C_n \subset C_{n+1}$ for each $n \in \mathbf{N}$, then $\bigcup_{n=1}^{\infty} C_n$ is connected.

(c) Deduce that, if $(C_n)_{n \in \mathbf{N}}$ is a sequence of connected subspaces of a topological space such that $C_n \cap C_{n+1} \neq \varnothing$ for each $n \in \mathbf{N}$, then $\bigcup_{n=1}^{\infty} C_n$ is connected.

[Hint: Use a mathematical induction argument on the sequence (D_n), where $D_n = \bigcup_{i=1}^{n} C_i$, and then apply the result of part (b) to (D_n).]

SAQ 21 Show that the unit circle

$$S^1 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = 1\}$$

in $\mathbf{R}^2$ is not homeomorphic with any interval in $\mathbf{R}$, where $\mathbf{R}^2$ and $\mathbf{R}$ both have their usual topologies.

SAQ 22 (a) Prove that, if $f: T_1 \to T_2$ is continuous and D is a component of the topological space T_2, then $f^{-1}(D)$ is a union of components of the topological space T_1.

[Hint: For any $x \in f^{-1}(D)$, let C_x be the component of T_1 that contains x. Show that $f(C_x) \subset D$, and deduce that $f^{-1}(D) = \bigcup_{x \in f^{-1}(D)} C_x$.]

(b) Give an example of explicit f, T_1, T_2 and D (as described in part (a)) such that $f^{-1}(D)$ is the union of two components of T_1. (You need not justify your assertions.)

SAQ 23 In Proposition 2, a particular equivalence relation on a topological space T was defined. The equivalence classes with respect to that relation are called *path-components* of T.

(i) Prove that each path-component of T is path-connected.

[Hint: Show that if $c \in T$ lies on a path in T between a and b then there is a path in T between a and c.]

(ii) Deduce that each component of T is a union of path-components of T.

(iii) Is it true that each path-component of T is necessarily closed in T?

SOLUTION TO SAQ 18 (a) The non-empty proper open sets of T are the intervals of the form $(-\infty, a)$ where $a \in \mathbf{R}$. Thus the non-empty proper closed sets of T are the intervals of the form $[a, \infty)$, $a \in \mathbf{R}$. Therefore there is no non-empty proper subset of T which is both open and closed in T. It follows that T is connected.

(b) It is shown in Example 9 on page 104 of *Unit 4* that the topology $\mathscr{T}$ is finer than the usual topology for $\mathbf{R}$. Thus the intervals $(-\infty, 0)$ and $(0, \infty)$ are open in T.

Also,

$$[0, \infty) = [0, 1) \cup (0, \infty)$$

is open in T. Thus, if we take

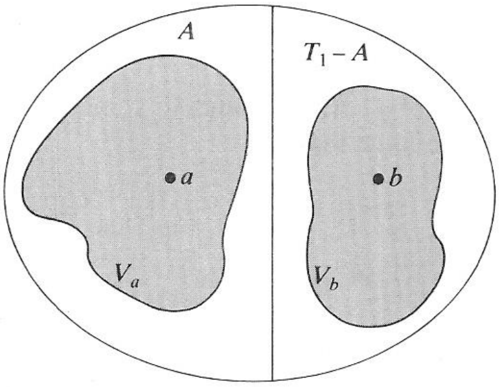
$$A = (-\infty, 0) \quad \text{and} \quad B = [0, \infty),$$

then $A \mid B$ is a partition of T. So T is not connected.

(c) The non-empty open sets of T are all subsets of $\mathbf{R}$ which contain 0. Hence the proper closed sets of T are all subsets of $\mathbf{R}$ which do not contain 0.

Thus there are no subsets of T, other than $\varnothing$ and T, which are both open and closed in T. Hence T is connected. ■

SOLUTION TO SAQ 19 Let $c \in f(T_1)$ and put $A = f^{-1}(c)$. Then A is a non-empty subset of T_1; and we shall prove that, in fact, $A = T_1$, by showing that A is both open and closed in T_1.



If $a \in A$, then, by hypothesis, there exists an open set V_a such that $a \in V_a$ and $f \mid V_a$ is constant. But $f(a) = c$ so that $f(x) = c$ for each x in V_a, and so $V_a \subset A$. Thus for each a in A, there exists an open set V_a of T_1 such that $a \in V_a \subset A$. It follows from SAQ 4 on page 97 of *Unit 4* that A is open in T_1.

Next, if $b \in T_1 - A$ then $f(b) \neq c$. By hypothesis, there exists V_b open in T_1 such that $b \in V_b$ and $f \mid V_b$ is constant. Thus for each x in V_b we have $f(x) = f(b) \neq c$, so that $V_b \subset T_1 - A$. It follows that $T_1 - A$ is open in T_1; and so A is closed in T_1.

Thus A is a non-empty subset of T_1 which is both open and closed in T_1. Since T_1 is a connected space, it follows that $A = T_1$.

Thus $f(x) = c$ for all x in T_1, and so f is a constant map. ■

SOLUTION TO SAQ 20

(a) Since $C \cap H$ is a non-empty set which is open and closed in the connected subspace C, it follows from **S** Corollary 6.2.4 that $C \cap H = C$. Therefore $C \subset H$.

(b) This follows at once from **S** Proposition 6.2.15.

(c) For each positive integer n, put $D_n = \bigcup_{i=1}^{n} C_i$. Since $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} C_n$ and $D_n \subset D_{n+1}$ for every n, the required result will follow from part (b) if we can show that D_n is connected for every n.

We may prove this by mathematical induction. Firstly, $D_1 = C_1$, which is connected. Suppose next that, for some k, D_k is connected. Since $D_{k+1} = D_k \cup C_{k+1}$, C_{k+1} is connected and $C_{k+1} \cap D_k \neq \varnothing$, it follows from **S** Proposition 6.2.15 that D_{k+1} is connected.

Hence, by the Principle of Mathematical Induction, D_n is connected for every positive integer n. ■

SOLUTION TO SAQ 21

(a) Here we use the technique of Note (viii) of Section 7.4.

If x is an interior point of an interval I in $\mathbf{R}$, then the subspace $I - \{x\}$ has two components. But if p is any point whatsoever of S^1 then the subspace $S^1 - \{p\}$ is path-connected and hence connected (i.e. has one component).

Therefore S^1 and I are not homeomorphic, in view of the result of Note (viii) of Section 7.4. ■

SOLUTION TO SAQ 22

(a) Choose any point $x \in f^{-1}(D)$, and let C_x be the component of T_1 which contains x. Then $f(x) \in f(C_x)$, which is connected (by **S** Proposition 6.2.10) since C_x is connected. Since $f(x) \in D$ and D is a component (i.e. a maximal connected subset) of T_2, it follows that $f(C_x) \subset D$.

By Proposition 4 on page 18 of *Unit 1*, it follows that

$$C_x \subset f^{-1}(D);$$

hence

$$\bigcup_{x \in f^{-1}(D)} C_x \subset f^{-1}(D). \qquad (1)$$

Since every point x of $f^{-1}(D)$ belongs to some component C_x on the left-hand side of (1), we have that

$$f^{-1}(D) \subset \bigcup_{x \in f^{-1}(D)} C_x;$$

and so $f^{-1}(D) = \bigcup_{x \in f^{-1}(D)} C_x$.

(b) Let $T_1 = (-2, -1) \cup (1, 2)$ and $T_2 = (1, 4)$ with the topologies induced by the usual topology on $\mathbf{R}$. Thus T_1 has two components $(-2, -1)$ and $(1, 2)$; and T_2 has one component $D = (1, 4)$.

Let $f: T_1 \to T_2$ be given by $f(x) = x^2$. Clearly, f is onto and is continuous, and $f^{-1}(D)$ $(= T_1)$ is the union of the two components of T_1. ■

SOLUTION TO SAQ 23

(i) Let D be a path-component of T. We must show that, if a, b are points of T, then there is a path in D from a to b.

By definition of $\equiv$, there is a path $f\colon [0,1] \to T$ from a to b in T. To complete the proof all we have to check is that $f([0,1]) \subset D$; for then $g\colon [0,1] \to D$ given by $g(t) = f(t)$, $t \in [0,1]$, is a path in D from a to b.

So let $c \in f([0,1])$, and let $f(u) = c$ for some $u \in [0,1]$. Now define a path $h\colon [0,1] \to T$ by

$$h(t) = f(ut), \qquad t \in [0,1].$$

Then h is continuous, $h(0) = f(0) = a$ and $h(1) = f(u) = c$. Thus $a \equiv c$, and so $c \in D$.

Therefore D is path-connected.

(ii) Let C be a component of T, and choose any point $x \in C$. If D_x is the path-component of T which contains x, then (by **S** Proposition 6.4.1) D_x is a connected subspace of T containing x; and so $D_x \subset C$.

Hence $\bigcup_{x \in C} D_x \subset C$; and since every point of C belongs to at least one of the D_x's, we have that

$$\bigcup_{x \in C} D_x = C.$$

That is, C is a union of path-components of T.

(iii) No. For example, the path-components of the subspace $G \cup J$ of $\mathbf{R}^2$ considered in **S** Examples 6.2.19 and 6.3.5 are G and J. Since $Cl(G) = G \cup J \neq G$, we see that G is not closed in $G \cup J$. ■

UNIT 8

COMPLETENESS

8.0 Introduction

In this unit we return to the study of metric spaces initiated in *Unit 3*. We have two main objectives: to give a characterization of compactness in metric spaces and to present some significant applications of metric space theory in analysis. You will find that, in meeting these objectives, we use most of the ideas that have been introduced in the option.

Cauchy's Convergence Criterion, that a sequence of real numbers converges if and only if it is a Cauchy sequence, is a major result in real analysis. It is closely related to the completeness property of **R** (**S** Axiom 1.1.4). In Section 8.1, we look at Cauchy sequences in an arbitrary metric space and find that there are metric spaces which contain non-convergent Cauchy sequences. Those metric spaces which share with **R** (with the Euclidean metric) the property that every Cauchy sequence is convergent, form a very important class. These are the *complete* metric spaces, which we study in Section 8.4.

In the introduction to *Unit 6*, we motivated the definition of compactness by considering properties of bounded closed intervals of **R**. There is one further property of such intervals which we might have examined: every sequence in a bounded closed interval I contains a subsequence which converges to a point of I. This property motivates the definition of *sequential compactness* in Section 8.2. We find that a metric space is compact if and only if it is sequentially compact. Ideas introduced in this section, together with *completeness*, are the essential ingredients for the description of compactness of a metric space using metric properties, in Section 8.5.

Applications of metric space theory in analysis often involve the complete metric space $\mathbf{C}[a,b]$; that is, the set $\mathscr{C}[a,b]$ of continuous real-valued functions on the closed interval $[a,b]$ with the 'sup metric' d_∞ you met in *Unit 3*. $\mathbf{C}[a,b]$ does not have the property, enjoyed by Euclidean spaces, that every bounded closed subset is compact. The Arzelà–Ascoli Theorem, which characterizes the compact subsets of $\mathbf{C}[a,b]$, is proved in Section 8.5.

Baire's Theorem for complete metric spaces, which we meet in Section 8.6, is one of the most important and useful theorems in topology, and has extensive applications in analysis. The section concludes with one of the most striking of these.

NOTE *Unit 8* contains some deep mathematical results and is written in a more condensed style than the other units. We suggest that *first* you skim through it to get the general drift, and *only then* read the details of the proofs. *Not* all of the unit is assessed; check the current Guide to the Examination for precise details.

8.1 Cauchy Sequences

Here we remind you of the definition of convergence of a sequence in a metric space, and then introduce the idea of a Cauchy sequence in a metric space. This notion occurs in the passage from **S** that we shall ask you to read in Section 8.2, but we have extracted it for separate study in this section because of its central role in the whole unit.

The definition of convergence of a sequence in a metric space occurred in Section 5.5 of *Unit 5*.

S *Definition 4.1.1* A sequence (x_n) of points in a metric space M with metric d *converges* to a point x of M if, given any (real number) $\varepsilon > 0$, there exists (an integer) N such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$.

This was the generalization to an arbitrary metric space of the familiar idea of convergence of a sequence of real numbers, **S** Definition 1.2.2, page 6. A sequence of real numbers converges in this sense precisely when it converges in $\mathbf{R}$ with the Euclidean metric.

Remarks

(i) Let (x_n) be a sequence in a metric space $M = \{A, d\}$, and let $x \in M$. Convergence of (x_n) to x in M is a *topological property* in the sense that, if (x_n) converges to x in M and d' is a metric on A which is topologically equivalent to d, then (x_n) converges to x in the metric space $M' = \{A, d'\}$.

To see this, let $\varepsilon > 0$ be given. Since d and d' are topologically equivalent, by Note (ii) on page 85 of *Unit 3*, there exists $\delta > 0$ such that

$$B_\delta(x; d) \subset B_\varepsilon(x; d').$$

Since (x_n) converges to x in M, there exists an N such that $x_n \in B_\delta(x; d)$ for all $n \geq N$; and so $x_n \in B_\varepsilon(x; d')$ for all $n \geq N$. Therefore (x_n) converges to x in $M' = \{A, d'\}$.

(ii) In the rest of this unit, we shall often be concerned with convergence of sequences that lie in subspaces of a given metric space.

Let H be a subspace of a metric space $M = \{A, d\}$. Recall that, by **S** Example 2.2.5, page 24, the induced metric d_H on H is given by

$$d_H(x, y) = d(x, y), \qquad x, y \in H.$$

Now let (x_n) be a sequence of points of H. We see that, if (x_n) converges in H to a point x of H, then (x_n) converges to x in M. Also, if (x_n) converges in M to a point x of H, then (x_n) converges to x in H.

However, if (x_n) converges in M to a point x of $M - H$, then (x_n) does *not* converge in H. For suppose that, in fact, (x_n) did converge in H to some point y of H. Then (x_n) would converge in M to two distinct points x, y of M; this is impossible since, by SAQ 19 on page 83 of *Unit 3*, every metric space is a Hausdorff space, and **S** Proposition 4.2.2 shows that a convergent sequence in a Hausdorff space has a *unique* limit.

EXAMPLE 1 Let M be $\mathbf{R}$ with the Euclidean metric and let H be the subspace $(0, 1]$ of M. For each positive integer n, put

$$x_n = 1 - \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

The sequence (x_n) converges to 1 in both M and H, whilst the sequence (y_n) converges to 0 in M, but does *not* converge in H since $0 \notin H$. ☐

In *Unit 2* (Note (ii) on page 57) we showed that a function $f: \mathbf{R} \to \mathbf{R}$ is continuous at $a \in \mathbf{R}$ if and only if, for each sequence (x_n) converging to a, the sequence $(f(x_n))$ converges to $f(a)$. We now generalize this to metric spaces.

ROPOSITION 1 Let $M_1 = \{A_1, d_1\}$ and $M_2 = \{A_2, d_2\}$ be metric spaces. Then a function $f: A_1 \to A_2$ is (d_1, d_2)-continuous at $a \in A_1$ if and only if, for each sequence (x_n) that converges to a in M_1, the sequence $(f(x_n))$ converges to $f(a)$ in M_2.

PROOF Suppose first that $f: A_1 \to A_2$ is (d_1, d_2)-continuous at $a \in A_1$ and let (x_n) be a sequence that converges to a in M_1. We must show that the sequence $(f(x_n))$ converges to $f(a)$ in M_2.

Let any $\varepsilon > 0$ be given. Since f is (d_1, d_2)-continuous at a, then (by **S** Definition 2.3.6, page 33) there exists $\delta > 0$ such that $f(B_\delta(a)) \subset B_\varepsilon(f(a))$. Since (x_n) converges to a in M_1, there exists a positive integer N such that $x_n \in B_\delta(a)$ for all $n \geq N$. Then $f(x_n) \in B_\varepsilon(f(a))$ for all $n \geq N$. Therefore $(f(x_n))$ converges to $f(a)$ in M_2, as required.

The proof of the converse is a little more subtle. (Before studying it, you might find it helpful to look again at Notes (iv) and (vi) on pages 51 and 52 of *Unit 2*.) We must show that f is (d_1, d_2)-continuous at a if for each sequence (x_n) that converges to a in M_1, the sequence $(f(x_n))$ converges to $f(a)$ in M_2. This will be achieved by showing that if f is *not* (d_1, d_2)-continuous at a, then *there exists* a sequence (x_n) which converges to a in M_1 such that $(f(x_n))$ does *not* converge to $f(a)$ in M_2.

If f is not (d_1, d_2)-continuous at a, then there exists a positive number ε_0 such that, for every $\delta > 0$, there exists some x in A_1 such that $d_1(x, a) < \delta$ and $d_2(f(x), f(a)) \geq \varepsilon_0$. In particular, taking $\delta = \frac{1}{n}$, we find that for each positive integer n there exists a point x_n of A_1 such that $d_1(x_n, a) < \frac{1}{n}$ and $d_2(f(x_n), f(a)) \geq \varepsilon_0$. The sequence (x_n) converges to a in M_1; for, given $\varepsilon > 0$, we can choose a positive integer N satisfying $\frac{1}{N} \leq \varepsilon$, so that $d_1(x_n, a) < \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$ for all $n \geq N$ and hence $x_n \in B_\varepsilon(a)$ for all $n \geq N$. But, as required, the sequence $(f(x_n))$ does not converge to $f(a)$ in M_2, for $d_2(f(x_n), f(a)) \geq \varepsilon_0$ for every integer n and so the ball $B_{\varepsilon_0}(f(a))$ contains no terms of the sequence $(f(x_n))$. ■

Cauchy sequences of real numbers were defined in **S** Chapter 1 (Definition 1.2.8, page 9). The following definition of a Cauchy sequence in an arbitrary metric space occurs in **S** Chapter 7, page 109, and is the obvious generalization to metric spaces of the notion of a Cauchy sequence of real numbers.

S *Definition 7.2.3* A sequence (x_n) in a metric space M with metric d is a *Cauchy sequence* if, given $\varepsilon > 0$, there exists N such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

It was shown in **S** Theorem 1.2.9, page 9, that a sequence of real numbers converges if and only if it is a Cauchy sequence. One half of this result extends to all metric spaces as follows.

PROPOSITION 2 Any convergent sequence in a metric space $M = \{A, d\}$ is a Cauchy sequence in M.

PROOF Suppose that (x_n) is a sequence in M which converges to a point x of M, and le $\varepsilon > 0$ be given. There exists a positive integer N such that $x_n \in B_{\frac{1}{2}\varepsilon}(x)$ for all $n \geq N$. Thus if $m, n \geq N$ then $d(x, x_n) < \frac{1}{2}\varepsilon$ and $d(x, x_m) < \frac{1}{2}\varepsilon$ so that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \qquad \text{(by Axiom (M3))} \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Hence (x_n) is a Cauchy sequence in M. ■

Remark. Proposition 2 supplies us with a plentiful stock of examples of Cauchy sequences in an arbitrary metric space M. In particular, let us say that a sequence (x_n) in a metric space M is *eventually constant* if there exists a positive integer N and some x in M such that $x_n = x$ for all $n \geq N$. If (x_n) is eventually constant and $x_n = x$ for all $n \geq N$, then (x_n) converges to x in M; and so (x_n) is a Cauchy sequence in M.

EXAMPLE 2 The converse of Proposition 2 is false.

For example, let H be the subspace $(0, 1]$ of $\mathbf{R}$ with the Euclidean metric, and for each positive integer n, put $y_n = \frac{1}{n}$. Since (y_n) is a Cauchy sequence of real numbers, it is a Cauchy sequence in H. But we saw in Example 1 that (y_n) does not converge in H. Thus (y_n) is a non-convergent Cauchy sequence in H. ☐

We have seen that convergence of a sequence in a metric space is a topological property. However, being a Cauchy sequence is a *metric-dependent* property in the following sense: If (x_n) is a Cauchy sequence in a metric space $M = \{A, d\}$ and d' is a metric on A which is topologically equivalent to d, then (x_n) need not be a Cauchy sequence in the metric space $M' = \{A, d'\}$. Of course, (x_n) must be a non-convergent Cauchy sequence in M for this to arise. Let us illustrate this phenomenon.

EXAMPLE 3 Let $M = \{\mathbf{R}, d\}$, where d is the metric on $\mathbf{R}$ (topologically equivalent to the Euclidean metric) given by

$$d(x, y) = \left|\frac{x}{1 + |x|} - \frac{y}{1 + |y|}\right|, \qquad x, y \in \mathbf{R}.$$

To see that d is a metric on $\mathbf{R}$ equivalent to the Euclidean metric, recall that in SAQ 23 on page 87 of *Unit 3* we showed that $x \mapsto x/(1 + |x|)$, $x \in \mathbf{R}$, is a homeomorphism of $\mathbf{R}$ with the Euclidean metric to the open interval $(-1, 1)$ with the Euclidean metric. The method of Example 18 on page 23 of *Unit 5* now shows that d is a metric on $\mathbf{R}$ topologically equivalent to the Euclidean metric.

Now for each positive integer n, put $x_n = n$. Then (x_n) is a Cauchy sequence in M. For if $n > m$, then

$$\begin{aligned} d(x_n, x_m) &= \left|\frac{n}{1 + n} - \frac{m}{1 + m}\right| \\ &= \frac{n - m}{(1 + n)(1 + m)} < \frac{n}{(1 + n)(1 + m)} < \frac{1}{1 + m}. \end{aligned}$$

Thus given $\varepsilon > 0$, choose a positive integer N such that $\frac{1}{1 + N} \leq \varepsilon$. If $n > m \geq N$, then

$$d(x_n, x_m) < \frac{1}{1 + m} \leq \frac{1}{1 + N} \leq \varepsilon;$$

hence (x_n) is a Cauchy sequence in M.

Since (x_n) is not convergent in $\mathbf{R}$ with the Euclidean metric, it is not convergent in M. Thus (x_n) is a (non-convergent) Cauchy sequence in M which is not a Cauchy sequence in $\mathbf{R}$ with the Euclidean metric. ☐

In Proposition 1, we saw that the continuous image of a convergent sequence is a convergent sequence. We shall see, in Example 4 below, that the continuous image of a (non-convergent) Cauchy sequence need not be a Cauchy sequence. The uniformly continuous image of a Cauchy sequence is, however, always a Cauchy sequence.

OPOSITION 3 If $M_1 = \{A_1, d_1\}$, $M_2 = \{A_2, d_2\}$ are metric spaces, $f: M_1 \to M_2$ is a uniformly continuous function and (x_n) is a Cauchy sequence in M_1, then $(f(x_n))$ is a Cauchy sequence in M_2.

PROOF Let $\varepsilon > 0$ be given. Since f is uniformly continuous, by S Definition 5.8.1, page 87, there exists $\delta > 0$ such that $d_2(f(x), f(y)) < \varepsilon$ for all x, y in M_1 such that $d_1(x, y) < \delta$. Since (x_n) is a Cauchy sequence in M_1, there exists a positive integer N such that $d_1(x_n, x_m) < \delta$ for all $m, n \geq N$. Thus $d_2(f(x_n), f(x_m)) < \varepsilon$ for all $m, n \geq N$. Hence $(f(x_n))$ is a Cauchy sequence in M_2. ■

EXAMPLE 4 Let $f: (0, 1] \to \mathbf{R}$ be given by

$$f(x) = \frac{1}{x}, \qquad 0 < x \leq 1.$$

If we denote **R** with the Euclidean metric by M, and the subspace $(0,1]$ of M by H, then $f\colon H \to M$ is continuous. Now for each positive integer n, put $y_n = \dfrac{1}{n}$. Then (y_n) is a Cauchy sequence in H; but $f(y_n) = n$ so that $(f(y_n))$ is not a Cauchy sequence in M.

Remark. Using Proposition 3, this gives us another proof that the continuous function $f\colon H \to M$ defined above is not uniformly continuous (see Example 13 on page 50 of *Unit 6*).

SAQ 1 Give an example of (a) a convergent Cauchy sequence, and (b) a non-convergent Cauchy sequence, both in the subspace $\mathbf{R} - \mathbf{Q}$ of **R** with the Euclidean metric.

SAQ 2 Let M be a discrete metric space (**S** Example 2.2.2, page 23). Show that

(a) the convergent sequences in M are precisely the eventually constant sequences; and

(b) the Cauchy sequences in M are precisely the eventually constant sequences.

SAQ 3 Let $M = \{\mathscr{C}[0,1], d_1\}$, where $\mathscr{C}[0,1]$ denotes the set of continuous real-valued functions on $[0,1]$ and

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx, \qquad f,g \in \mathscr{C}[0,1]$$

(see **S** Example 2.2.9, page 28). Show that if

$$f_n(x) = \begin{cases} \operatorname{Min}\,\{n, x^{-1/2}\}, & \text{if } x \in (0,1], \\ n, & \text{if } x = 0, \end{cases}$$

then (f_n) is a Cauchy sequence in M.

SAQ 4 Let $M = \{\mathbf{R}^2, e\}$ where e is the 'lollipop metric' for $\mathbf{R}^2$ introduced in Example 2, page 70, of *Unit 3*.

(a) Prove that each Cauchy sequence in M is a Cauchy sequence in $\mathbf{R}^2$ with the Euclidean metric d_2.

(b) Show that if $y_n = \left(\dfrac{1}{n}, 1\right)$ for each n in **N**, then (y_n) is a Cauchy sequence in $\{\mathbf{R}^2, d_2\}$, but not a Cauchy sequence in M.

***SAQ 5** Let (x_n), (y_n) be Cauchy sequences in a metric space M with metric d; put $a_n = d(x_n, y_n)$. Show that (a_n) is a Cauchy sequence of real numbers.

[Hint: You might find the result of SAQ 4 of *Unit 3* useful.]

SOLUTION TO SAQ 1

(a) Any sequence of irrationals which converges in **R** to an irrational number will do, e.g. the sequence $\left(\sqrt{2} + \dfrac{1}{n}\right)$ converges to $\sqrt{2}$ and so is a convergent Cauchy sequence in $\mathbf{R} - \mathbf{Q}$.

(b) We should look for a sequence of irrational numbers which converges in **R** to a rational number. For example, for each positive integer n, let

$$x_n = \frac{1}{n}\sqrt{2}.$$

Then x_n is an irrational number and the sequence (x_n) converges in **R** to the rational number 0. Thus (x_n) is a non-convergent Cauchy sequence in $\mathbf{R} - \mathbf{Q}$. ■

SOLUTION TO SAQ 2

(a) Let (x_n) be a sequence in M which converges to x. Then there exists a positive integer N such that $x_n \in B_1(x)$ for all $n \geq N$. But $B_1(x) = \{x\}$, so that $x_n = x$ for all $n \geq N$; and so the sequence (x_n) is eventually constant. In the opposite direction, as we observed in the Remark before Example 2, an eventually constant sequence is convergent.

(b) Let (x_n) be a sequence in M which is not eventually constant. Then for each positive integer N there exist m, $n \geq N$ such that $x_n \neq x_m$ and so $d(x_n, x_m) = 1$. Thus there exists no positive integer N such that $d(x_n, x_m) < 1$ for all m, $n \geq N$ and so (x_n) is not a Cauchy sequence in M. Hence a Cauchy sequence is necessarily eventually constant.

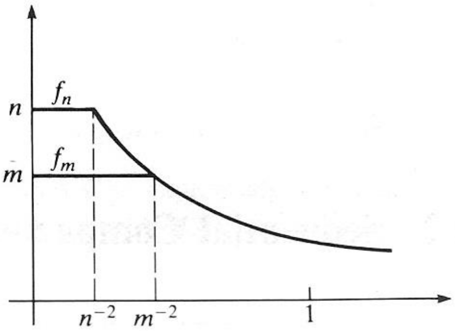
Conversely, as we saw in the Remark before Example 2, an eventually constant sequence is a Cauchy sequence. ■

SOLUTION TO SAQ 3 To show that (f_n) is a Cauchy sequence in M, we must consider $d_1(f_n, f_m)$. Suppose that m, n are positive integers such that $n > m$. Then (by looking at $[0,1]$ as the union of $[0, n^{-2}]$, $[n^{-2}, m^{-2}]$ and $[m^{-2}, 1]$) we have that

$$d_1(f_n, f_m) = \int_0^{n^{-2}} (n - m)dx + \int_{n^{-2}}^{m^{-2}} (x^{-1/2} - m)dx$$

$$= \frac{n-m}{n^2} + \left(\frac{2}{m} - \frac{2}{n}\right) - m\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

$$= \frac{1}{m} - \frac{1}{n} < \frac{1}{m}.$$



Now let $\varepsilon > 0$ be given, and choose a positive integer N such that $\frac{1}{N} \leq \varepsilon$. If $n > m \geq N$ then

$$d_1(f_n, f_m) < \frac{1}{m} \leq \frac{1}{N} \leq \varepsilon.$$

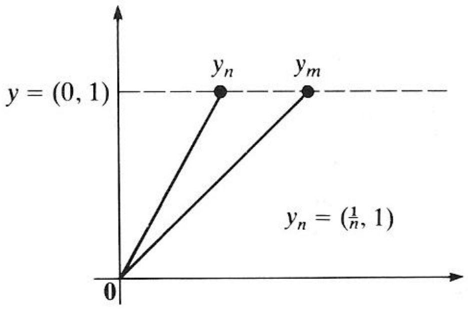
Thus (f_n) is a Cauchy sequence in M. ■

SOLUTION TO SAQ 4 (a) Suppose that (x_n) is a Cauchy sequence in M and let $\varepsilon > 0$ be given. Then there exists a positive integer N such that $e(x_n, x_m) < \varepsilon$ for all m, $n \geq N$. But, since (as we saw in Example 11 of *Unit 3*, page 82)

$$B_\varepsilon(a; e) \subset B_\varepsilon(a; d_2) \qquad \text{(for each } a \in \mathbf{R}^2, \varepsilon > 0\text{)}$$

we have that $d_2(x, y) \leq e(x, y)$ for all x, y in $\mathbf{R}^2$. Thus $d_2(x_n, x_m) < \varepsilon$ for all m, $n \geq N$; and so (x_n) is a Cauchy sequence in $\{\mathbf{R}^2, d_2\}$.

(b) The sequence (y_n) converges in $\{\mathbf{R}^2, d_2\}$ to $y = (0, 1)$. Hence (y_n) is a Cauchy sequence in $\{\mathbf{R}^2, d_2\}$, by Proposition 2.



But if $n \in \mathbf{N}$, then $d_2(\mathbf{0}, y_n) > 1$ so that if m, $n \in \mathbf{N}$ and $m \neq n$ then

$$e(y_n, y_m) = d_2(\mathbf{0}, y_n) + d_2(\mathbf{0}, y_m) > 2.$$

Thus $e(y_n, y_m) > 2$ for all m, n in $\mathbf{N}$ with $m \neq n$; and so (y_n) is not a Cauchy sequence in M. ■

SOLUTION TO SAQ 5 We must consider

$$|a_n - a_m| = |d(x_n, y_n) - d(x_m, y_m)|.$$

Using an old trick (again!), we write

$$a_n - a_m = d(x_n, y_n) - d(x_m, y_n) + d(x_m, y_n) - d(x_m, y_m).$$

Hence

$$\begin{aligned}|a_n - a_m| &\le |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)|\\ &\le d(x_n, x_m) + d(y_n, y_m),\end{aligned}$$

the second inequality being obtained by applying SAQ 4 on page 71 of *Unit 3* to each of the two expressions.

Now let $\varepsilon > 0$ be given. Since (x_n) is a Cauchy sequence, there exists a positive integer N_1 such that $d(x_n, x_m) < \frac{1}{2}\varepsilon$ for all $m, n \ge N_1$. Also, there exists a positive integer N_2 such that $d(y_n, y_m) < \frac{1}{2}\varepsilon$ for all $m, n \ge N_2$. If we put $N = \text{Max}\{N_1, N_2\}$, then for all $m, n \ge N$ we have

$$|a_n - a_m| \le d(x_n, x_m) + d(y_n, y_m) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Hence (a_n) is a Cauchy sequence of real numbers. ■

8.2 Sequential Compactness

READ S from the beginning of Chapter 7 on page 108, to page 110, line −12.

NOTES

(i) *Page 108, Definition 7.2.1*
If (x_n) is a sequence in a metric space M and $(n(1), n(2), \dots, n(k), \dots)$ is a sequence of positive integers such that $n(k) < n(k+1)$ for every k, then

$$(x_{n(1)}, x_{n(2)}, \dots, x_{n(k)}, \dots)$$

is a sequence in M. The *subsequences of the sequence* (x_n) are *all* sequences in M obtained in this way, for *different* choices of the sequence $(n(k))$.

For example, if $x_n = (-1)^n$, $n \in \mathbf{N}$, then we obtain a subsequence of the sequence (x_n) in $\mathbf{R}$ by taking $n(k) = 2k$ for each k; since $x_{n(k)} = (-1)^{2k} = 1$ for every k, this subsequence is the constant sequence with value 1.

(ii) *Page 108, Lemma 7.2.2*
This result was essentially established in SAQ 29 of *Unit 5*.

(iii) *Page 109, Proposition 7.2.4*
This result was stated and proved as Proposition 2 of Section 8.1 of this unit.

(iv) *Page 109, Lemma 7.2.5*
Note the distinction between the *limit points* of S and the *limit* of (x_n). The limit of (x_n), if it exists, is certainly a limit point of S. But S can have many limit points, even when (x_n) has no limit. For example, consider the sequence $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \dots\}$. It has no limit but the corresponding set S has two limit points 0 and 1. Clearly, the subsequences $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ and $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$ converge to 0 and 1 respectively.

(v) *Page 110, lines 12 to 21*
The Bolzano–Weierstrass Theorem (SAQ 8 of *Unit 6*, page 37) states that the infinite subset S of the compact space C has a limit point in C.

(vi) *Page 110, lines −14, −13*
Let us show that if S is a bounded subset of a metric space $M = \{A, d\}$, then $Cl(S)$ is a bounded subset of M.

By **S** Definition 2.2.12, page 29, there exists a in A and a positive real number K such that $d(x, a) \leq K$ for all x in S. Now by **S** Proposition 3.7.12, page 61, if $y \in Cl(S)$ then there exists x in S such that $d(y, x) < 1$, so that

$$d(y, a) \leq d(y, x) + d(x, a) < K + 1.$$

Therefore $d(y, a) < K + 1$ for all y in $Cl(S)$ and so $Cl(S)$ is a bounded subset of M. ▲

SAQ 6 Let $M = \{\mathbf{R}^2, e\}$, where e is the 'lollipop metric' introduced in Example 2 of *Unit 3*, page 70.

(a) Show that $(1, 1)$ is a point of the closure in M of the set

$$C = \{(x_1, x_2) \in \mathbf{R}^2 \colon 1 < x_2 \leq x_1\}.$$

(b) Find a sequence in C which converges in M to $(1, 1)$.

SAQ 7 Prove that if a sequence in a metric space M converges to a point x of M, then every subsequence of (x_n) converges to x.

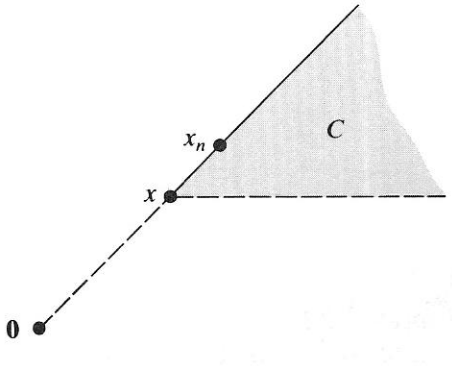
***SAQ 8** Prove that if C is a subset of a metric space M, and C is sequentially compact in itself, then C is closed in M. [Hint: Use the result of SAQ 7 and **S** Lemma 7.2.2.]

SOLUTION TO SAQ 6

(a) Put $x = (1, 1)$. We must show that every open ball of M, centre x, contains a point of C. (You might find it helpful to look at the description of the open balls of M in Example 11 of *Unit 3*, page 82.)

For each positive integer n, put $x_n = \left(1 + \frac{1}{n}, 1 + \frac{1}{n}\right)$ and note that x_n is a point of C. Then $e(x, x_n)$ is equal to the Euclidean distance $d_2(x, x_n) = \frac{\sqrt{2}}{n}$.

If $\varepsilon > 0$, we can choose a positive integer N such that $\frac{\sqrt{2}}{N} < \varepsilon$. Then $e(x, x_N) < \varepsilon$ so that $x_N \in B_\varepsilon(x)$. Hence for every $\varepsilon > 0$, the open ball $B_\varepsilon(x)$ in M contains a point of C. Hence $x \in Cl(C)$.



(b) Clearly the sequence (x_n) in C, defined in the first part of the solution, converges in M to x. Any other sequence in C which lies on the line $x_1 = x_2$ and converges to x in $\mathbf{R}^2$ with the Euclidean metric would serve equally well. ■

SOLUTION TO SAQ 7 Let $\varepsilon > 0$ be given. Since (x_n) converges to x in M, there exists a positive integer N such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. Now let $(x_{n(k)})$ be a subsequence of (x_n) where $(n(k))$ is a strictly increasing sequence of positive integers. There exists a positive integer K such that $n(K) \geq N$. If $k \geq K$, then $n(k) \geq n(K) \geq N$. Therefore $x_{n(k)} \in B_\varepsilon(x)$ for all $k \geq K$. Hence the subsequence $(x_{n(k)})$ converges to x. ■

SOLUTION TO SAQ 8 If $x \in Cl(C)$ then by **S** Lemma 7.2.2 there is a sequence (x_n) in C which converges to x. Since C is sequentially compact in itself, by **S** Definition 7.2.1, the sequence (x_n) in C contains some subsequence which converges to a point of C. But, by SAQ 7, every subsequence of (x_n) converges to x. Therefore $x \in C$. Thus $Cl(C) = C$, and so C is closed in M. ■

In the next reading passage, we complete the proof that sequential compactness is equivalent to compactness in metric spaces.

READ S page 110, line -8, to page 112, line 18.

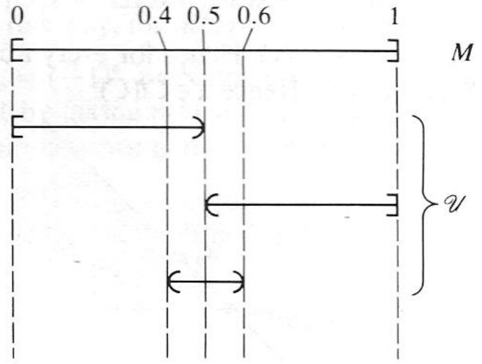
NOTES

(i) *Page 111, lines 15 to 19*
Total boundedness will be studied in more detail in Section 8.5. We shall see that every totally bounded metric space is bounded, and that there exist bounded metric spaces that are not totally bounded.

(ii) *Page 111, Definition 7.2.10*
If a metric space M is totally bounded, then for each $\varepsilon > 0$, it has a finite ε-net, i.e. a finite subset S of M such that $M = \bigcup_{x \in S} B_\varepsilon(x)$. Of course, the set S depends on ε. The smaller ε, the larger we should expect S to be.

Also, if $\varepsilon > 0$ then an ε-net for a metric space is also an ε'-net for each $\varepsilon' > \varepsilon$.

(iii) *Page 111, Definition 7.2.11*
The Lebesgue number is the radius of an open ball which, if centred anywhere in the space M, will be entirely within *one* of the sets in the cover. For example, if $M = [0, 1]$ with the usual metric and $\mathscr{U} = \{[0, 0.5), (0.5, 1], (0.4, 0.6)\}$, then the Lebesgue number for $\mathscr{U}$ is 0.1.



(iv) *Page 112, Theorem 7.2.13*
It is a consequence of this theorem and **S** Proposition 7.2.6 that *a metric space is compact if and only if it is sequentially compact.* We can now see, from **S** Proposition 7.2.9 and **S** Definition 7.2.10, that *every compact metric space is totally bounded.* ▲

SAQ 9 (a) Find a finite 10^{-3}-net for the subspace $(0, 1)$ of $\mathbf{R}$ with the Euclidean metric.

(b) Give an example of a totally bounded metric space which is not compact.

SAQ 10 Let M denote the subspace $[0, 1]$ of $\mathbf{R}$, with the Euclidean metric. Find a Lebesgue number for the open cover of M which consists of the open balls of M, centre the points $\frac{r}{10}$, $r = 0, 1, \ldots, 10$, and radius $\frac{1}{10}$.

OLUTION TO SAQ 9 (a) For $n = 1, \dots, 999$, put $x_n = n \cdot 10^{-3}$. Since $B_{10^{-3}}(x_n) = (x_{n-1}, x_{n+1})$ for each n (where $x_0 = 0$ and $x_{1000} = 1$), we see that $\{x_1, x_2, \dots, x_{999}\}$ is a 10^{-3}-net for $(0, 1)$ with the Euclidean metric.

(b) By an obvious generalization of the argument for part (a), we can show that the subspace $(0, 1)$ of $\mathbf{R}$, with the Euclidean metric, is totally bounded. We must show that a finite ε-net exists for every $\varepsilon > 0$.

Let $\varepsilon > 0$ be given, and choose a positive integer N such that $\frac{1}{N} \leq \varepsilon$. Now, for $n = 1, 2, \dots, N-1$, put $x_n = \frac{n}{N}$. For each of these n, $(x_{n-1}, x_{n+1}) \subset B_\varepsilon(x_n)$ (where $x_0 = 0$, $x_N = 1$), and so we see that $\{x_1, \dots, x_{N-1}\}$ is a finite ε-net for $(0, 1)$ with the Euclidean metric.

The subspace $(0, 1)$ of $\mathbf{R}$ with the Euclidean metric is not compact, so we have the required example. ■

OLUTION TO SAQ 10 For each $r = 0, 1, \dots, 10$, denote the open ball in M with centre $\frac{r}{10}$ and radius $\frac{1}{10}$ by B_r and for each x in $[0, 1]$ denote the open ball in M with centre x and radius $\frac{1}{20}$ by $C(x)$. We shall find a Lebesgue number for the cover $\{B_0, \dots, B_{10}\}$ of M by showing that, for each x, there is some r such that $C(x) \subset B_r$.

Observe that if $0 \leq x < \frac{1}{20}$ then $C(x) \subset B_0$. Also, if $r = 1, \dots, 9$, and

$$\frac{2r-1}{20} \leq x < \frac{2r+1}{20},$$

then $C(x) \subset B_r$. Finally, if $\frac{19}{20} \leq x \leq 1$, then $C(x) \subset B_{10}$.

It follows that $\frac{1}{20}$ is a Lebesgue number for the given cover. Of course, any number ε such that $0 < \varepsilon \leq \frac{1}{20}$ is *also* a Lebesgue number for the cover. ■

8.3 Function Spaces

In this section, we study the metric space $\{\mathscr{B}([a,b],\mathbf{R}), d_\infty\}$ and its subspace $\{\mathscr{C}[a,b], d_\infty\}$. Recall (see **S** Example 2.2.8, page 27) that $\mathscr{B}([a,b],\mathbf{R})$ denotes the set of bounded functions $f\colon [a,b] \to \mathbf{R}$ and d_∞ is the metric on $\mathscr{B}([a,b],\mathbf{R})$ defined by

$$d_\infty(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|, \qquad f, g \in \mathscr{B}([a,b],\mathbf{R}).$$

Also, $\mathscr{C}[a,b]$ is the subset of $\mathscr{B}([a,b],\mathbf{R})$ consisting of the continuous functions $f\colon [a,b] \to \mathbf{R}$. We denote the metric space $\{\mathscr{B}([a,b],\mathbf{R}), \mathrm{d}_\infty\}$ by $\mathbf{B}[a,b]$ and its metric subspace $\{\mathscr{C}[a,b], d_\infty\}$ by $\mathbf{C}[a,b]$, for brevity.

AMPLE 5 Suppose f, g in $\mathscr{C}[0,1]$ are defined by

$$f(x) = \frac{1}{x+3}, \quad g(x) = \frac{1}{x+6}, \qquad x \in [0,1];$$

let us find $d_\infty(f,g)$.

If $x \in [0, 1]$, then

$$|f(x) - g(x)| = \frac{1}{x+3} - \frac{1}{x+6} = \frac{3}{(x+3)(x+6)}.$$

Now $(x + 3)(x + 6) \geq 18$ if $0 \leq x \leq 1$; and we see that the supremum of $|f(x) - g(x)|$ is attained when $x = 0$. Thus

$$d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)| = |f(0) - g(0)| = \frac{1}{6}.$$

Remark. This example illustrates the fact that, if $f, g \in \mathscr{C}[a, b]$, then there exists some c in $[a, b]$ such that

$$d_\infty(f, g) = |f(c) - g(c)|.$$

For, the real-valued function on $[a, b]$ given by

$$x \mapsto |f(x) - g(x)|, \quad x \in [a, b],$$

is continuous; and so by **S** Corollary 5.5.4, page 84, it is bounded and attains its bounds. Hence there exists some c in $[a, b]$ such that

$$|f(c) - g(c)| = \sup_{x \in [a,b]} |f(x) - g(x)| = d_\infty(f, g). \quad \square$$

Next we consider open balls in $\mathbf{B}[a, b]$, $\mathbf{C}[a, b]$. Let $\varepsilon > 0$ be given. Let $f \in \mathbf{B}[a, b]$ and $g \in B_\varepsilon(f)$, so that $d_\infty(f, g) < \varepsilon$. Thus

$$\sup_{x \in [a,b]} |f(x) - g(x)| < \varepsilon,$$

and so

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in [a, b].$$

In the opposite direction, suppose that f, g are any pair of elements of $\mathbf{B}[a, b]$ such that

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in [a, b].$$

Then it follows that

$$d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| \leq \varepsilon.$$

(Note that '$<$' has become '$\leq$' here.)

In the case of $\mathbf{C}[a, b]$, on the other hand, we have (the more expected result):

PROPOSITION 4 Let $f \in \mathscr{C}[a, b]$ and $\varepsilon > 0$. Then, in $\mathbf{C}[a, b]$, $B_\varepsilon(f)$ consists of all those functions g in $\mathscr{C}[a, b]$ such that

$$|f(x) - g(x)| < \varepsilon$$

for all x in $[a, b]$.

PROOF First, we note that, if $g \in \mathscr{C}[a, b]$, then (by definition of B_ε and d_∞) $g \in B_\varepsilon(f)$ if and only if $d_\infty(f, g) < \varepsilon$. As we saw above, if $g \in \mathscr{C}[a, b]$ and $d_\infty(f, g) < \varepsilon$, then

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \text{ in } [a, b].$$

Conversely, suppose that $g \in \mathscr{C}[a, b]$ and that

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \text{ in } [a, b].$$

Then, by the Remark above, there exists some c in $[a, b]$ such that

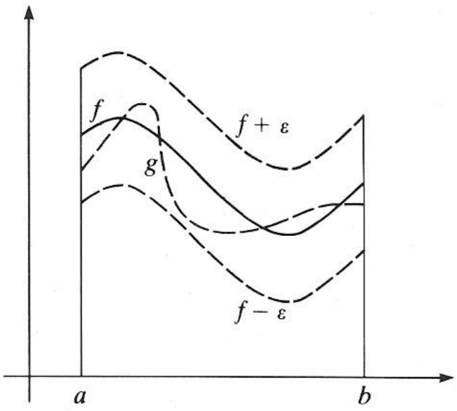
$$d_\infty(f, g) = |f(c) - g(c)|;$$

and so $d_\infty(f, g) < \varepsilon$. $\blacksquare$

Remark. It follows from Proposition 4 that if $f, g \in \mathscr{C}[a, b]$ and $\varepsilon > 0$, then $g \in B_\varepsilon(f)$ if and only if

$$f(x) - \varepsilon < g(x) < f(x) + \varepsilon$$

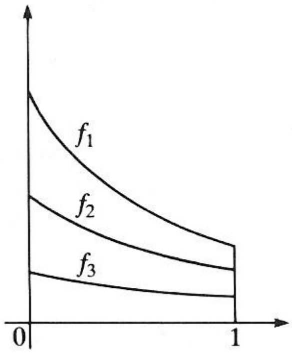
for all x in $[a, b]$. Thus, if we draw the 'band' of width 2ε centred on the graph of f (see diagram), then $g \in B_\varepsilon(f)$ if and only if the graph of g lies within this band.

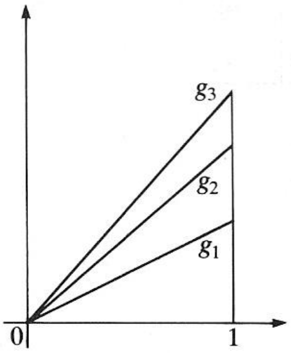


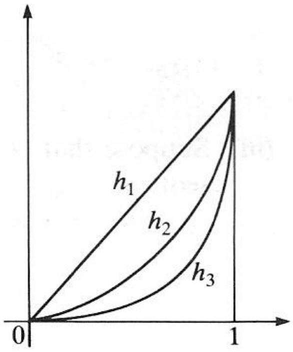
Our primary interest in this section is in Cauchy sequences in the metric spaces $\mathbf{B}[a, b]$ and $\mathbf{C}[a, b]$, and the convergence of sequences in these spaces. We first look at some examples of sequences in $\mathbf{C}[a, b]$.

EXAMPLE 6 For each positive integer n, let f_n, g_n, h_n be the continuous real-valued functions defined on $[0, 1]$ by

$$f_n(x) = \frac{1}{x+n}, \quad g_n(x) = nx, \quad h_n(x) = x^n, \qquad x \in [0, 1].$$







Then (f_n), (g_n) and (h_n) are sequences in $\mathbf{C}[a, b]$. We shall discuss their convergence properties later in the section. ☐

Convergence in $\mathbf{B}[a, b]$ can be described as follows.

OPOSITION 5 A sequence (f_n) converges in $\mathbf{B}[a, b]$ to f if and only if the following condition is satisfied:

for each $\varepsilon > 0$ there exists a positive integer N such that

$$|f(x) - f_n(x)| < \varepsilon \qquad (*)$$

for all $n \geq N$ and all x in $[a, b]$.

Remark. The key point to note in $(*)$ is that once ε is chosen, we can find an N (independent of x) such that the inequality holds for all $n \geq N$ and all x.

PROOF First, we suppose that (f_n) converges to f in $\mathbf{B}[a, b]$, and we show that condition $(*)$ is satisfied. So let $\varepsilon > 0$ be given. Since (f_n) converges to f, there exists a positive integer N such that $d_\infty(f, f_n) < \varepsilon$ for all $n \geq N$. Now as we saw above, if $d_\infty(f, f_n) < \varepsilon$ then

$$|f(x) - f_n(x)| < \varepsilon$$

for all x in $[a, b]$. Thus condition $(*)$ is satisfied.

Conversely, suppose that condition $(*)$ is satisfied, and let $\varepsilon > 0$ be given. Then there exists a positive integer N such that

$$|f(x) - f_n(x)| < \tfrac{1}{2}\varepsilon$$

for all x in $[a, b]$ and all $n \geq N$. Thus for all $n \geq N$ we have

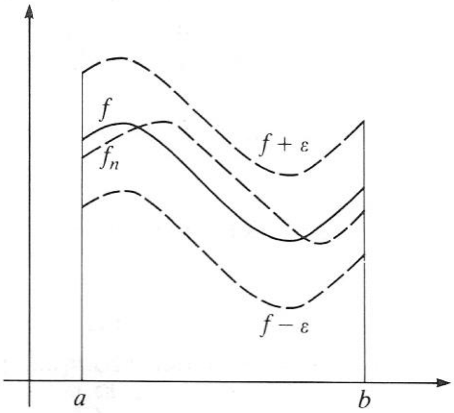
$$d_\infty(f, f_n) \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

Hence (f_n) converges to f in $\mathbf{B}[a, b]$. ■

Remarks

(i) Since $\mathbf{C}[a, b]$ is a subspace of $\mathbf{B}[a, b]$, we have a similar characterization of convergence in $\mathbf{C}[a, b]$.

(ii) Convergence in $\mathbf{B}[a, b]$ can be described geometrically as follows. Suppose that the sequence (f_n) converges to f in $\mathbf{B}[a, b]$, and let ε be an arbitrary positive number. If we draw the 'band' of width 2ε centred on the graph of f (see the diagram), then there exists a positive integer N (depending on ε) such that for all $n \geq N$ the graph of f_n lies within this band.



(iii) Suppose that (f_n) converges to f in $\mathbf{B}[a, b]$. Then for fixed c in $[a, b]$, the sequence $(f_n(c))$ of real numbers converges (in $\mathbf{R}$) to the real number $f(c)$. For let $\varepsilon > 0$ be given. Then there exists a positive integer N such that

$$|f(x) - f_n(x)| < \varepsilon$$

for all x in $[a, b]$ and all $n \geq N$. In particular,

$$|f(c) - f_n(c)| < \varepsilon \qquad \text{for all } n \geq N.$$

This is precisely what we need to establish that

$$\lim_{n \to \infty} f_n(c) = f(c).$$

Thus *if* (f_n) *converges to* f *in* $\mathbf{B}[a, b]$ *then* $\lim\limits_{n \to \infty} f_n(x) = f(x)$ *for all* x *in* $[a, b]$.

However, the converse of this result is *false*. More precisely, if (f_n) is a sequence in $\mathscr{B}([a, b], \mathbf{R})$, $f \in \mathscr{B}([a, b], \mathbf{R})$ and $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for all x in $[a, b]$, then it is not necessarily the case that (f_n) converges to f in $\mathbf{B}[a, b]$. A counter-example will be found in Example 8 below.

But first let us see why convergence of (f_n) can run into difficulty.

Let $\varepsilon > 0$ be given. For each x in $[a, b]$, $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$; so there exists a positive integer N_x, depending on x as well as ε, such that

$$|f(x) - f_n(x)| < \varepsilon \qquad \text{for all } n \geq N_x.$$

But for (f_n) to converge to f in $\mathbf{B}[a, b]$, we would require an integer N, depending *only* on ε, such that

$$|f(x) - f_n(x)| < \varepsilon$$

for all x in $[a, b]$ and all $n \geq N$. If there exists an integer N such that $N \geq N_x$ for all x in $[a, b]$, then this condition will be satisfied; but we cannot expect, in general, that such an integer N will exist.

Convergence of the type described by (∗) is so important in analysis that we give it a special name.

FINITION A sequence (f_n) of functions *converges uniformly* to a function f on a subset A of $\mathbf{R}$ if: for each $\varepsilon > 0$, there exists a positive integer N depending only on ε such that

$$|f(x) - f_n(x)| < \varepsilon \qquad \text{for all } n \geq N \text{ and all } x \in A.$$

Then we can rephrase the result of Proposition 5 as: in $\mathbf{B}[a, b]$, convergence and uniform convergence are equivalent.

FINITION A sequence (f_n) of functions *converges pointwise* to a function f on a subset A of $\mathbf{R}$ if: for each $\varepsilon > 0$ and each $x \in A$ there exists a positive integer N such that

$$|f(x) - f_n(x)| < \varepsilon \qquad \text{for all } n \geq N.$$

The first two paragraphs of Remark (iii) explain that in $\mathbf{B}[a, b]$, convergence implies pointwise convergence, but pointwise convergence does not imply convergence.

XAMPLE 7 Let us test the sequences (f_n), (g_n) in Example 6 for convergence in $\mathbf{B}[0, 1]$.

We show first that (g_n) does not converge in $\mathbf{B}[0, 1]$. If (g_n) were convergent in $\mathbf{B}[0, 1]$, then by Remark (iii) (above), $(g_n(x))$ would be a convergent sequence of real numbers for each x in $[0, 1]$. But $g_n(1) = n$ for each n in $\mathbf{N}$ so that $(g_n(1))$ is not convergent. Hence (g_n) does not converge in $\mathbf{B}[0, 1]$.

The sequence (f_n) does converge in $\mathbf{B}[0, 1]$. Since

$$f_n(x) = \frac{1}{n + x}, \quad x \in [0, 1],$$

we see that, for each $x \in [0, 1]$,

$$f_n(x) = \frac{1/n}{1 + x/n} \longrightarrow \frac{0}{1} = 0 \quad \text{as } n \longrightarrow \infty.$$

Hence if we denote by $\mathbf{0}$ the zero function on $[0, 1]$ given by $\mathbf{0}(x) = 0$, $x \in [0, 1]$, (f_n) converges pointwise to $\mathbf{0}$.

If (f_n) is convergent in $\mathbf{B}[0, 1]$ then, by Remark (iii) preceding this example, it must converge to $\mathbf{0}$. We shall show that this is the case.

We must consider $|f_n(x) - \mathbf{0}(x)|$ for x in $[0, 1]$. Now

$$|f_n(x) - \mathbf{0}(x)| = f_n(x) = \frac{1}{n + x} \leq \frac{1}{n}$$

for all x in $[0, 1]$.

Let $\varepsilon > 0$ be given, and choose a positive integer N such that $\frac{1}{N} < \varepsilon$. For all x in $[0, 1]$ and all $n \geq N$ we have

$$|f_n(x) - \mathbf{0}(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence (f_n) converges to $\mathbf{0}$ in $\mathbf{B}[0, 1]$, by Proposition 5. ☐

Notice that in the preceding example we have a sequence (f_n) of continuous functions converging in $\mathbf{B}[0, 1]$ to the zero function, which, of course, is continuous on $[0, 1]$. This is an instance of a general result which we now prove.

POSITION **6** If (f_n) converges to f in $\mathbf{B}[a, b]$ and each f_n is continuous, then f is continuous.

PROOF Let c be an arbitrary point of $[a,b]$; we must show that f is continuous at c. Since (f_n) converges to f in $\mathbf{B}[a,b]$, given $\varepsilon > 0$, there exists a positive integer N such that

$$|f(x) - f_n(x)| < \varepsilon$$

for all x in $[a,b]$ and all $n \geq N$. In particular,

$$|f(x) - f_N(x)| < \varepsilon \qquad \text{for all } x \in [a,b]$$

and

$$|f(c) - f_N(c)| < \varepsilon.$$

Now by hypothesis f_N is continuous at c; hence there exists $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \varepsilon$$

for all x in $[a,b]$ such that $|x - c| < \delta$. Since

$$f(x) - f(c) = \{f(x) - f_N(x)\} + \{f_N(x) - f_N(c)\} + \{f_N(c) - f(c)\}$$

(where we have played a standard trick!), we see that for all x in $[a,b]$ such that $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Thus f is continuous at c, as required. ■

COROLLARY The subspace $\mathbf{C}[a,b]$ is closed in $\mathbf{B}[a,b]$.

PROOF Let K denote the closure of $\mathscr{C}[a,b]$ in $\mathbf{B}[a,b]$. If $f \in K$ then, by **S** Lemma 7.2.2, there exists a sequence (f_n) in $\mathscr{C}[a,b]$ which converges in $\mathbf{B}[a,b]$ to f. Since $f_n \in \mathscr{C}[a,b]$ for each n, it follows from Proposition 6 that $f \in \mathscr{C}[a,b]$. Thus $K = \mathscr{C}[a,b]$ and so $\mathscr{C}[a,b]$ is a closed subset of $\mathbf{B}[a,b]$. ■

EXAMPLE 8 We can now present the counterexample promised in Remark (iii) following Proposition 5.

Let (h_n) be the sequence given by

$$h_n(x) = x^n, \quad x \in [0,1].$$

If $0 \leq x < 1$, then $x^n \longrightarrow 0$ as $n \longrightarrow \infty$. Also, $h_n(1) = 1$ for all n, so that $h_n(1) \longrightarrow 1$ as $n \longrightarrow \infty$. If we take h to be the function defined by

$$h(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1, \end{cases}$$

then $h_n(x) \longrightarrow h(x)$ as $n \longrightarrow \infty$, for each $x \in [0,1]$.

Also, h is a bounded function and so is in $\mathbf{B}[0,1]$. But (h_n) does not converge to h in $\mathbf{B}[0,1]$; for if it did, then as each h_n is continuous, the limit h would be continuous (by Proposition 6). Clearly, h is not continuous at 1. Hence (h_n) is not convergent in $\mathbf{B}[0,1]$. □

Cauchy sequences in $\mathbf{B}[a,b]$ have the following characterization.

PROPOSITION 7 A sequence (f_n) is a Cauchy sequence in $\mathbf{B}[a,b]$ if and only if for each $\varepsilon > 0$ there exists a positive integer N such that

$$|f_n(x) - f_m(x)| < \varepsilon$$

for all x in $[a,b]$ and all $m, n \geq N$.

PROOF Suppose first that (f_n) is a Cauchy sequence in $\mathbf{B}[a,b]$. Then by **S** Definition 7.2.3, given $\varepsilon > 0$ there exists N such that $d_\infty(f_n, f_m) < \varepsilon$ for all $m, n \geq N$. As established in the discussion preceding Proposition 4, it follows that

$$|f_n(x) - f_m(x)| < \varepsilon$$

for all x in $[a,b]$ and all $m, n \geq N$.

Conversely, suppose that the sequence (f_n) satisfies the stated condition. If $\varepsilon > 0$, there exists N such that

$$|f_n(x) - f_m(x)| < \tfrac{1}{2}\varepsilon$$

for all x in $[a, b]$ and all $m, n \geq N$. Thus for all $m, n \geq N$, we have $d_\infty(f_n, f_m) \leq \frac{1}{2}\varepsilon < \varepsilon$. Hence (f_n) is a Cauchy sequence in $\mathbf{B}[a, b]$. ■

Remarks

(i) Since $\mathbf{C}[a, b]$ is a subspace of $\mathbf{B}[a, b]$, we have a similar characterization of Cauchy sequences in $\mathbf{B}[a, b]$.

(ii) Let (f_n) be a Cauchy sequence in $\mathbf{B}[a, b]$. It is an immediate consequence of Proposition 7 that, for each x in $[a, b]$, the sequence $(f_n(x))$ of real numbers is a Cauchy sequence in $\mathbf{R}$ (in the sense of **S** Definition 1.2.8).

We are now ready for the main result of this section.

POSITION **8** A sequence is convergent in $\mathbf{B}[a, b]$ if and only if it is a Cauchy sequence in $\mathbf{B}[a, b]$.

PROOF It follows from Proposition 2 that a convergent sequence in $\mathbf{B}[a, b]$ is a Cauchy sequence in $\mathbf{B}[a, b]$.

Now let (f_n) be a Cauchy sequence in $\mathbf{B}[a, b]$. We must show that (f_n) is convergent in $\mathbf{B}[a, b]$. Let $x \in [a, b]$. By Remark (ii) above, $(f_n(x))$ is a Cauchy sequence of real numbers. Hence by **S** Theorem 1.2.9, $(f_n(x))$ converges; let us denote its limit by $f(x)$. In this way we obtain a function $f\colon [a, b] \to \mathbf{R}$ such that

$$\lim_{n \to \infty} f_n(x) = f(x)$$

for each x in $[a, b]$. We shall complete the proof by showing that f is a bounded function and that (f_n) converges to f in $\mathbf{B}[a, b]$.

Let $\varepsilon > 0$ be given. Since (f_n) is a Cauchy sequence in $\mathbf{B}[a, b]$, by Proposition 7 there exists a positive integer N such that

$$|f_n(x) - f_m(x)| < \tfrac{1}{2}\varepsilon$$

for all x in $[a, b]$ and all $m, n \geq N$. Also, since $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$, for each x in $[a, b]$ there exists an integer $m(x)$ such that $m(x) \geq N$ and

$$|f(x) - f_{m(x)}(x)| < \tfrac{1}{2}\varepsilon.$$

Thus for all x in $[a, b]$ and all $n \geq N$ we have

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_{m(x)}(x)| + |f_{m(x)}(x) - f_n(x)| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Summarizing, we have proved:

> for each $\varepsilon > 0$, there exists a positive integer N such that
>
> $$|f(x) - f_n(x)| < \varepsilon \qquad (*)$$
>
> for all x in $[a, b]$ and all $n \geq N$.

By Proposition 5, if f is a bounded function, then (f_n) converges to f in $\mathbf{B}[a, b]$. Hence it remains to show only that f is bounded.

Taking $\varepsilon = 1$ in $(*)$, we see that there exists a positive integer k such that

$$|f(x) - f_k(x)| < 1 \qquad \text{for all } x \text{ in } [a, b].$$

Thus

$$f_k(x) - 1 < f(x) < f_k(x) + 1$$

for all x in $[a, b]$. But f_k is a bounded function, as there exist real numbers m_k, M_k such that $m_k \leq f(x) \leq M_k$ for all x in $[a, b]$. Therefore

$$m_k - 1 < f(x) < M_k + 1$$

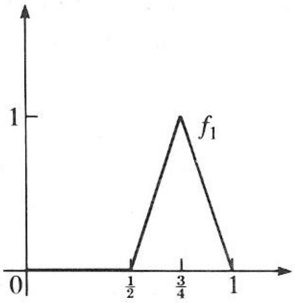
for all x in $[a, b]$. Thus f is a bounded function, as required. ■

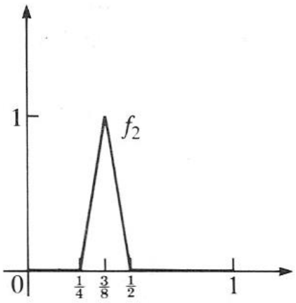
We conclude with an example of a bounded closed subset of the metric space $\mathbf{C}[0,1]$ that is not compact.

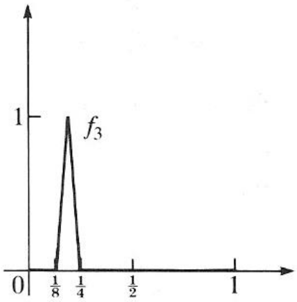
EXAMPLE 9 For each positive integer n, let f_n be the element of $\mathbf{C}[0,1]$ given by

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 2^{-n}, \\ 2^{n+1}x - 2, & \text{if } 2^{-n} \leq x \leq 3 \cdot 2^{-(n+1)}, \\ 4 - 2^{n+1}x, & \text{if } 3 \cdot 2^{-(n+1)} \leq x \leq 2^{-(n-1)}, \\ 0, & \text{if } 2^{-(n-1)} \leq x \leq 1. \end{cases}$$

The graphs of f_1, f_2, f_3 are shown below. For obvious reasons, (f_n) is known as the *moving bump*!







If $\mathbf{0}$ denotes the zero function on $[0,1]$ (i.e. $\mathbf{0}(x) = 0$, $x \in [0,1]$) then, for each n in $\mathbf{N}$,

$$d_\infty(\mathbf{0}, f_n) = f_n(3 \cdot 2^{-(n+1)}) = 1.$$

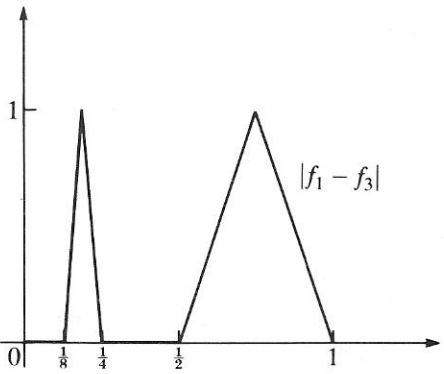
Thus if

$$K = \{f \in \mathscr{C}[0,1] : d_\infty(\mathbf{0}, f) \leq 1\}$$

then (f_n) is a sequence in K.

We are going to use the sequence (f_n) to show that the bounded closed subset K of $\mathbf{C}[0,1]$ is not compact, via a contradiction argument.

Suppose that K is compact. Then by **S** Proposition 7.2.6, it is sequentially compact. Thus the sequence (f_n) in K contains a convergent subsequence.



But no subsequence of (f_n) is a Cauchy sequence. For if m, n are distinct positive integers, then the graph of the function $x \mapsto |f_n(x) - f_m(x)|$ has two 'peaks'.(see the diagram) so that

$$d_\infty(f_n, f_m) = \sup_{x \in [0,1]} |f_n(x) - f_m(x)| = 1.$$

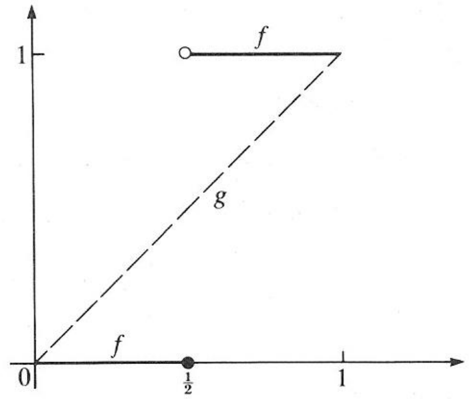
Since $d_\infty(f_n, f_m) = 1$ whenever m and n are distinct positive integers, we see that no subsequence of the sequence (f_n) can be a Cauchy sequence; and so, by Proposition 2, no subsequence of (f_n) is convergent.

This contradiction shows that K is not compact.

SAQ 11 Let f in $\mathbf{B}[0,1]$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \tfrac{1}{2}, \\ 1, & \text{if } \tfrac{1}{2} < x \leq 1, \end{cases}$$

and let $g(x) = x$, $x \in [0,1]$. Find $d_\infty(f, g)$.



SAQ 12 For each positive integer n, let f_n in $\mathbf{C}[0,1]$ be defined by

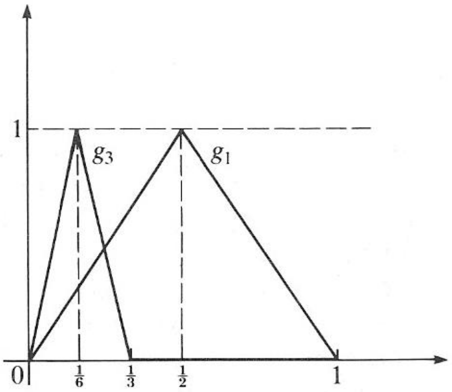
$$f_n(x) = \frac{x}{1+nx}, \quad x \in [0,1].$$

(a) Show that $\lim_{n\to\infty} f_n(x) = 0$ for each x in $[0,1]$.

(b) Show that the sequence converges in $\mathbf{C}[0,1]$.

SAQ 13 For each positive integer n, let g_n in $\mathbf{C}[0,1]$ be defined by

$$g_n(x) = \begin{cases} 2nx, & \text{if } 0 \leq x \leq \dfrac{1}{2n}, \\ 2 - 2nx, & \text{if } \dfrac{1}{2n} \leq x \leq \dfrac{1}{n}, \\ 0, & \text{if } \dfrac{1}{n} \leq x \leq 1. \end{cases}$$



Show that the sequence (g_n) does not converge in $\mathbf{C}[0,1]$.

SOLUTION TO SAQ 11 By considering the graph of the function

$$x \mapsto |f(x) - g(x)|, \quad x \in [0,1],$$

we see that

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| = g(\tfrac{1}{2}) - f(\tfrac{1}{2}) = \tfrac{1}{2}. \quad \blacksquare$$

SOLUTION TO SAQ 12 (a) If $0 < x \leq 1$ then

$$f_n(x) = \frac{x}{1+nx} = \frac{x/n}{(1/n)+x} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Also $f_n(0) = 0$ for every n. Thus $f_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for every x in $[0,1]$.

(b) We know from part (a) that, if (f_n) converges in $\mathbf{C}[0,1]$, then it must converge to the zero function $\mathbf{0}$. Now if $0 < x \leq 1$ then

$$|f_n(x) - \mathbf{0}(x)| = f_n(x) = \frac{1}{x^{-1}+n} \leq \frac{1}{n+1}.$$

Also, $f_n(0) = 0$ for every n. Thus $|f_n(x) - \mathbf{0}(x)| \leq (n+1)^{-1}$ for every x in $[0,1]$.

Now let $\varepsilon > 0$ be given, and choose a positive integer N such that $(N+1)^{-1} < \varepsilon$. For all x in $[0,1]$ and all $n \geq N$ we have

$$\begin{aligned}|f_n(x) - \mathbf{0}(x)| &\leq (n+1)^{-1}\\ &\leq (N+1)^{-1} < \varepsilon.\end{aligned}$$

Thus (f_n) converges to $\mathbf{0}$ in $\mathbf{C}[0,1]$. ■

SOLUTION TO SAQ 13 First we observe that

$$\lim_{n\to\infty} g_n(x) = 0 \qquad \text{for every } x \text{ in } [0,1].$$

For $g_n(0) = 0$ for all n in $\mathbf{N}$. Also, if x is a real number such that $0 < x \leq 1$ then $g_n(x) = 0$ for every n such that $\frac{1}{n} \leq x$. Thus given x such that $0 < x \leq 1$, we choose a positive integer N such that $\frac{1}{N} \leq x$. Then for all $n \geq N$, $g_n(x) = 0$, and hence $g_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$.

We know that if (g_n) converges in $\mathbf{C}[0,1]$, it must converge to the zero function $\mathbf{0}$. But it is clear that for every positive integer n,

$$d_\infty(\mathbf{0}, g_n) = g_n\left(\frac{1}{2n}\right) = 1.$$

Thus g_n does not converge to $\mathbf{0}$ in $\mathbf{C}[0,1]$, and hence (g_n) does not converge in $\mathbf{C}[0,1]$. ■

8.4 Complete Metric Spaces

We saw in *Unit 2* that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. In Section 8.1 of this unit we defined Cauchy sequences in an arbitrary metric space and found that, whilst every convergent sequence is a Cauchy sequence, in some metric spaces there are non-convergent Cauchy sequences. The metric space $\mathbf{B}[a,b]$ has, however, the property that every Cauchy sequence which it contains is convergent. This property, that a sequence converges if and only if it is a Cauchy sequence, turns out to be one of the most significant in the theory of metric spaces and leads us to the definition of *completeness* for metric spaces.

READ S from the beginning of Chapter 9 on page 123, to page 125, line 13.

NOTE *Page 123, lines −8, −7*
S points out here that completeness is not a topological property, for $\mathbf{R}$ with the Euclidean metric is complete and homeomorphic with $(0,1)$ with the Euclidean metric, which is not complete. In Example 11 we shall meet topologically equivalent metrics such that one of the corresponding metric spaces is complete whilst the other is not complete. ▲

EXAMPLE 10 The metric space $\mathbf{B}[a,b]$ is complete, since every Cauchy sequence in $\mathbf{B}[a,b]$ converges, by Proposition 8 of Section 8.3. Also, by the Corollary to Proposition 6, the subspace $\mathbf{C}[a,b]$ is closed in $\mathbf{B}[a,b]$ so that, by **S** Proposition 9.2.3(b), the metric space $\mathbf{C}[a,b]$ is complete. ☐

The completeness of $\mathbf{C}[a,b]$ has many applications in analysis, some of which we shall present in Section 8.6.

XAMPLE 11 In Example 3, we saw that the metric d on $\mathbf{R}$ defined by

$$d(x, y) = \left|\frac{x}{1 + |x|} - \frac{y}{1 + |y|}\right|, \qquad x, y \in \mathbf{R},$$

is topologically equivalent to the Euclidean metric on $\mathbf{R}$.

Whilst $\mathbf{R}$ with the Euclidean metric is complete, $M = \{\mathbf{R}, d\}$ is not complete, since, as we found in Example 3, (n) is a non-convergent Cauchy sequence in M.

XAMPLE 12 The metric space $M = \{\mathbf{R}^2, e\}$, where e is the 'lollipop metric' on $\mathbf{R}^2$ introduced in Example 2 of *Unit 3*, is a complete metric space. The proof of this is quite hard, so omit this Example if you are short of time.

Let (x_n) be a Cauchy sequence in M. We must prove that (x_n) is convergent in M. We established in SAQ 4 that (x_n) is a Cauchy sequence in $\mathbf{R}^2$ with the Euclidean metric d_2. Since, by S Example 9.2.6, $\{\mathbf{R}^2, d_2\}$ is complete, it follows that the sequence (x_n) converges in $\{\mathbf{R}^2, d_2\}$, to x, say.

We shall complete the proof by showing that (x_n) converges to x in M.

Let $\varepsilon > 0$ be given. We must show that there exists a positive integer N such that $x_n \in B_\varepsilon(x; e)$ for $n \geq N$. We consider the two cases (a) $x = \mathbf{0}$ and (b) $x \neq \mathbf{0}$.

Case (a): $x = \mathbf{0}$

Since (x_n) converges to $\mathbf{0}$ in $\{\mathbf{R}^2, d_2\}$, there exists a positive integer N such that $x_n \in B_\varepsilon(\mathbf{0}; d_2)$ for all $n \geq N$. But, as shown in Example 11 on page 82 of *Unit 3*,

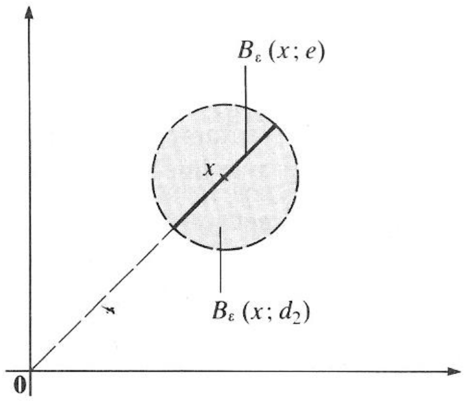
$$B_\varepsilon(\mathbf{0}; e) = B_\varepsilon(\mathbf{0}; d_2).$$

Thus $x_n \in B_\varepsilon(\mathbf{0}; e)$ for all $n \geq N$ and so (x_n) converges to $\mathbf{0}$ in M as required.

Case (b): $x \neq \mathbf{0}$

There is no loss of generality in supposing that

$$0 < \varepsilon < \tfrac{1}{2} d_2(\mathbf{0}, x).$$



The relationship between $B_\varepsilon(x; d_2)$ and $B_\varepsilon(x; e)$ is as shown in the diagram; that is, $B_\varepsilon(x; d_2)$ is the open disc centre x and radius ε, whilst $B_\varepsilon(x; e)$ consists of the points of the disc that lie on the diameter which, when extended, passes through the origin. We know that there exists a positive integer N_1 such that $x_n \in B_\varepsilon(x; d_2)$ for all $n \geq N_1$. We shall show that there exists $N \geq N_1$ such that $x_n \in B_\varepsilon(x; e)$ for all $n \geq N$.

We show first that there exists a positive integer $N \geq N_1$ such that x_n and x_N are collinear with the origin for all $n > N$. Since (x_n) is a Cauchy sequence in M, there exists a positive integer N_2 such that $e(x_n, x_m) < \varepsilon$ for all $m, n \geq N_2$. Now take $N = \operatorname{Max}\{N_1, N_2\}$.

If $n \geq N$, then, by (M3) for d_2, we have

$$d_2(\mathbf{0}, x_n) + d_2(x_n, x) \geq d_2(\mathbf{0}, x),$$

so that

$$d_2(\mathbf{0}, x_n) \geq d_2(\mathbf{0}, x) - d_2(x_n, x).$$

But $d_2(\mathbf{0}, x) > 2\varepsilon$ and $d_2(x_n, x) < \varepsilon$, since $n \geq N \geq N_1$, so that

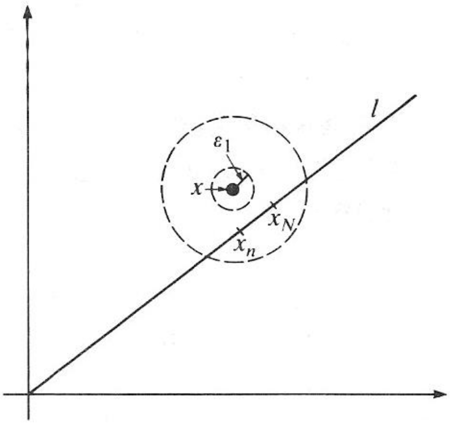
$$d_2(\mathbf{0}, x_n) > 2\varepsilon - \varepsilon = \varepsilon.$$

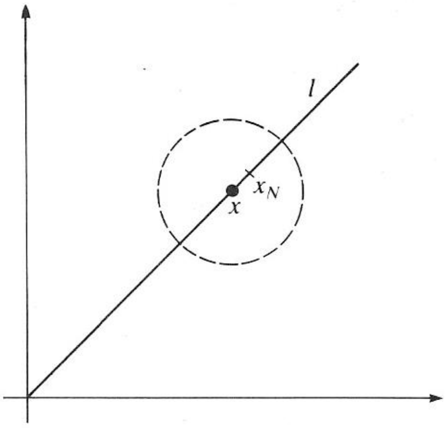
Now if $n > N$ and x_n, x_N are *not* collinear with the origin, then

$$e(x_n, x_N) = d_2(\mathbf{0}, x_n) + d_2(\mathbf{0}, x_N) > 2\varepsilon,$$

and this is a contradiction, since $e(x_n, x_N) < \varepsilon$ for n, $N \geq N_2$. Thus for all $n > N$, x_n lies on the line l which passes through x_N and the origin.

Now suppose that l is *distinct* from the line passing through x and the origin. Then there is an open ε_1-ball in $\{\mathbf{R}^2, d_2\}$ with centre x, which does not meet l; so that for all $n \geq N$ we have $x_n \notin B_{\varepsilon_1}(x)$, which contradicts the convergence of (x_n) to x in $\{\mathbf{R}^2, d_2\}$. Thus x lies on the line l.





Hence for all $n \geq N$, $x_n \in B_\varepsilon(x; d_2)$ and x_n, x are collinear with the origin. Therefore $x_n \in B_\varepsilon(x; e)$ for all $n \geq N$. Thus we see that (x_n) converges to x in M as required.

This completes the proof that M is a complete metric space.

SAQ 14 Prove that every discrete metric space is complete.

SAQ 15 Prove that Cantor's Ternary Set K (Example 12 of *Unit 6*) is a complete metric space. [Hint: The question has a 'one-line' solution.]

SAQ 16 Let d_1, d_2 be Lipschitz equivalent metrics (**S** Proposal 2.4.3) on a set A, and $M_1 = \{A, d_1\}$, $M_2 = \{A, d_2\}$ be metric spaces.

(a) Prove that every Cauchy sequence in M_2 is a Cauchy sequence in M_1.

(b) Prove that every convergent sequence in M_1 is a convergent sequence in M_2.

(c) Deduce that if M_1 is complete then so is M_2.

SAQ 17 Assume (without proof) that the function ρ defined on the set $\mathscr{C}[a, b]$ of continuous real-valued functions with domain the closed interval $[a, b]$ by

$$\rho(f, g) = \sup_{x \in [a,b]} \{e^{-k(x-a)} |f(x) - g(x)|\}, \qquad f, g \in \mathscr{C}[a, b],$$

where k is a positive constant, is a metric. Show that ρ is Lipschitz equivalent to the metric d_∞ on $\mathscr{C}[a, b]$, and deduce that $\{\mathscr{C}[a, b], \rho\}$ is a complete metric space.

SOLUTION TO SAQ 14 It was shown in SAQ 2 that the Cauchy sequences in a discrete metric space M are precisely the sequences in M which are eventually constant. Thus every Cauchy sequence in M is convergent, and so M is a complete metric space. ■

SOLUTION TO SAQ 15 K compact metric space (Example 12, *Unit 6*) $\Rightarrow$ K complete (**S** Proposition 9.2.4). ■

SOLUTION TO SAQ 16 Since d_1, d_2 are Lipschitz equivalent, there exist strictly positive constants h, k, such that

$$hd_2(x,y) \le d_1(x,y) \le kd_2(x,y) \qquad (*)$$

for all x, y in A.

(a) Let (x_n) be a Cauchy sequence in M_2, and let $\varepsilon > 0$ be given. Since $\varepsilon/k > 0$, there exists a positive integer N such that $d_2(x_n; x_m) < \varepsilon/k$ for all m, $n \ge N$. Therefore $d_1(x_n, x_m) < \varepsilon$ for all m, $n \ge N$.

Thus (x_n) is a Cauchy sequence in M_1.

(b) Let the sequence (x_n) converge to x in M_1, and let $\varepsilon > 0$ be given. Since $h\varepsilon > 0$, there exists a positive integer N such that $d_1(x, x_n) < h\varepsilon$ for all $n \ge N$. Therefore $d_2(x, x_n) < \varepsilon$ for all $n \ge N$. Thus (x_n) is a convergent sequence in M_2. (Note that we have rephrased Definition 4.1.1 of convergent sequences in terms of d_1 and d_2 rather than open balls, simply in order to make use of the inequalities $(*)$.)

(c) Let (x_n) be a Cauchy sequence in M_2. By (a), (x_n) is a Cauchy sequence in M_1, and so a convergent sequence in M_1 since M_1 is complete. By (b), (x_n) is then necessarily a convergent sequence in M_2, so that M_2 must be complete. ■

SOLUTION TO SAQ 17 Since $e^{-k(b-a)} \le e^{-k(x-a)} \le 1$ for all x in $[a,b]$, we see that

$$e^{-k(b-a)} d_\infty(f,g) \le \rho(f,g) \le d_\infty(f,g)$$

for all f, g in $\mathscr{C}[a,b]$. Thus the metrics ρ and d_∞ on $\mathscr{C}[a,b]$ are Lipschitz equivalent.

Since $\{\mathscr{C}[a,b], d_\infty\} = \mathbf{C}[a,b]$ is a complete metric space, it follows from SAQ 16 that $\{\mathscr{C}[a,b], \rho\}$ is a complete metric space. ■

8.5 Compactness in Metric Spaces

We have seen that a compact metric space is complete (S Proposition 9.2.4) and totally bounded (S Propositions 7.2.6 and 7.2.9). In this section, the characterization of compactness for metric spaces is completed by proving that every metric space which is complete and totally bounded is compact. Recall (S Definition 7.2.10) that a metric space M is *totally bounded* if it has a finite ε-net for every $\varepsilon > 0$, i.e. for every $\varepsilon > 0$, there exists a finite subset S of M such that $M \subset \bigcup_{x \in S} B_\varepsilon(x)$.

READ S from Lemma 10.1.1 on page 139, to the end of Section 10.1 on page 141.

NOTES (i) *Page 140, Corollary 10.1.2*
For the proof of this, note that any ε-net for the space will do as an ε-net for the subspace.

(ii) *Page 140, Corollary 10.1.2*
In SAQ 9 we proved directly that the subspace $(0,1)$ of $\mathbf{R}$ with the Euclidean metric is totally bounded. We can, alternatively, deduce this fact from Corollary 10.1.2, since $(0,1) \subset [0,1]$ and $[0,1]$ with the Euclidean metric is compact and so totally bounded.

(iii) *Page 140, Example 10.1.6*
The 'moving bump' sequence which **S** employs here is the sequence of functions constructed in Example 9.

(iv) *Page 141, Theorem 10.1.8*
Combining this result with **S** Propositions 9.2.4 and 7.2.9 we see that *a metric space is compact if and only if it is complete and totally bounded.* ▲

SAQ 18 (a) Let d be a metric on a set A and let $\bar{d}$ be the metric on A (see SAQ 3(ii), page 71, and Example 13, page 87, of *Unit 3*) given by

$$\bar{d}(x, y) = \mathrm{Min}\{1, d(x, y)\}, \qquad x, y \in A.$$

Prove that $\{A, d\}$ is totally bounded if and only if $\{A, \bar{d}\}$ is totally bounded.

(b) Show that $\mathbf{R}$ with the metric

$$\bar{d}(x, y) = \mathrm{Min}\{1, |x - y|\}, \qquad x, y \in \mathbf{R},$$

is bounded but not totally bounded.

SAQ 19 (Converse of Lemma 10.1.7) Prove that, if every sequence in a metric space M has a subsequence which is itself a Cauchy sequence in M, then M is totally bounded. [Hint: Look again at the proof of **S** Proposition 7.2.9.]

SOLUTION TO SAQ 18 (a) If $\varepsilon > 0$ then an ε-net for a metric space is an ε'-net for each $\varepsilon' > \varepsilon$. Thus to show that a metric space is totally bounded, it is enough to show that it has a finite ε-net for each ε such that $0 < \varepsilon \leq 1$.

Now consider the metric spaces $\{A, d\}$ and $\{A, \bar{d}\}$. If $x \in A$ and $0 < \varepsilon \leq 1$ then

$$B_\varepsilon(x; d) = B_\varepsilon(x; \bar{d}).$$

It is now clear that $\{A, d\}$ has a finite ε-net, where $0 < \varepsilon \leq 1$, if and only if $\{A, \bar{d}\}$ has a finite ε-net. Therefore $\{A, d\}$ is totally bounded if and only if $\{A, \bar{d}\}$ is totally bounded.

(b) Since $\mathbf{R}$ with the Euclidean metric is complete and not compact, it is not totally bounded. It follows from (a) that $\mathbf{R}$, with the metric $\bar{d}$, is not totally bounded.

Also, $\{\mathbf{R}, \bar{d}\}$ is bounded, since $\bar{d}(x, y) \leq 1$ for all x, y in $\mathbf{R}$. ■

Remark. It would *not* be correct to assert that the result of part (a) is a consequence of the fact (see Example 13 of *Unit 3*) that the metrics d and $\bar{d}$ on A are topologically equivalent. Total boundedness is *not* a topological property.

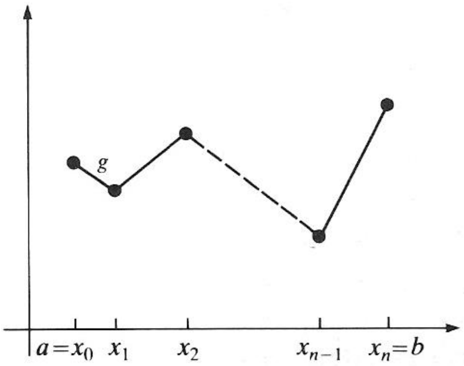
It can be shown, for example, that $\{\mathbf{R}, d\}$ is totally bounded, where d is the metric on $\mathbf{R}$ introduced in Example 3. The metric d is topologically equivalent to the Euclidean metric, but $\mathbf{R}$ with the Euclidean metric is not totally bounded.

SOLUTION TO SAQ 19 In the proof of **S** Proposition 7.2.9, it was shown that if M is a metric space with the property that, for some $\varepsilon > 0$ there is no finite ε-net for M, then there exists a sequence in M which has no Cauchy subsequence. We can now interpret this as follows: if M is a metric space which is not totally bounded, then there exists a sequence in M which has no Cauchy subsequence. Thus if every sequence in a metric space M has a Cauchy subsequence, then M must be totally bounded. ■

In Example 9, we saw that a bounded closed subset of $\mathbf{C}[a, b]$ is not necessarily compact. In the remainder of this section we shall employ the characterization of compactness in Note (iv) to describe those subsets of $\mathscr{C}[a, b]$ which are compact. This description is the content of the Arzelà–Ascoli Theorem, which we shall prove as Proposition 11. It turns out that the condition which a bounded closed subset of $\mathbf{C}[a, b]$ must satisfy to ensure that it is compact is *equicontinuity*, which we now define.

EFINITION A subset $\mathscr{F}$ of $\mathscr{C}[a,b]$ is said to be *equicontinuous* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for *all* f in $\mathscr{F}$ and *all* x, y in $[a,b]$ such that $|x - y| < \delta$.

Remark. Since, by **S** Proposition 5.8.2, each member of $\mathscr{C}[a,b]$ is uniformly continuous, it follows that if $\mathscr{F}$ is any subset of $\mathscr{C}[a,b]$, then given $\varepsilon > 0$, for each f in $\mathscr{F}$ there exists $\delta > 0$ (which depends on f and ε) such that $|f(x) - f(y)| < \varepsilon$ for all x, y in $[a,b]$ such that $|x - y| < \delta$. The force of the definition of equicontinuity is that if $\mathscr{F}$ is equicontinuous, then given $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) such that *for all* f in $\mathscr{F}$, $|f(x) - f(y)| < \varepsilon$ for all x, y in $[a,b]$ such that $|x - y| < \delta$ (that is, one δ suffices for all functions in $\mathscr{F}$).



An important part in the proof of the Arzelà–Ascoli Theorem and in the rest of our work on $\mathbf{C}[a,b]$ will be played by certain simple continuous functions on $[a,b]$, called *piecewise-linear functions*, whose graphs are polygonal paths in $\mathbf{R}^2$. More precisely, we shall say that g in $\mathscr{C}[a,b]$ is a piecewise-linear function if there exist $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the graph of g restricted to $[x_k, x_{k+1}]$ is the (straight) line-segment joining $(x_k, g(x_k))$ to $(x_{k+1}, g(x_{k+1}))$ for $k = 0, 1, \ldots, n-1$ (see the diagram). (You met some examples of piecewise-linear functions in the 'moving bump' sequence, Example 9.)

As a final preliminary to Proposition 9, which is the main step towards the Arzelà–Ascoli Theorem, note that a subset $\mathscr{F}$ of $\mathscr{C}[a,b]$ is bounded in the metric space $\mathbf{C}[a,b]$ if and only if there exists a positive real number M such that $d_\infty(\mathbf{0}, f) \le M$ for all f in $\mathscr{F}$, where $\mathbf{0}$ denotes the zero function $(x \mapsto 0, x \in [a,b])$. Thus $\mathscr{F}$ is bounded in $\mathbf{C}[a,b]$ if and only if there exists $M > 0$ such that

$$|f(x)| \le M$$

for *all* f in $\mathscr{F}$ and *all* x in $[a,b]$.

ROPOSITION **9** If $\mathscr{F}$ is a bounded equicontinuous subset of $\mathbf{C}[a,b]$, then $\mathscr{F}$ is totally bounded.

PROOF Let $\varepsilon > 0$ be given. We are going to construct a finite subset $\mathscr{G}$ of $\mathbf{C}[a,b]$ (in fact, consisting of piecewise-linear functions) such that for each f in $\mathscr{F}$ there exists at least one g in $\mathscr{G}$ such that $f \in B_\varepsilon(g)$. By **S** Lemma 10.1.1, this will be enough to show that $\mathscr{F}$ is totally bounded.

First, note that since $\mathscr{F}$ is bounded, there exists a positive real number M such that

$$-M \le f(x) \le M$$

for all x in $[a,b]$ and all f in $\mathscr{F}$. Thus the graph of each member of $\mathscr{F}$ lies in the rectangle $[a,b] \times [-M, M]$.

Next we use the equicontinuity of $\mathscr{F}$ to divide the rectangle $[a,b] \times [-M,M]$ into certain useful sub-rectangles. Since $\mathscr{F}$ is equicontinuous, there exists $\delta > 0$ such that for all f in $\mathscr{F}$,

$$|f(x) - f(y)| < \tfrac{1}{4}\varepsilon \qquad (1)$$

for all x, y in $[a,b]$ such that $|x - y| < \delta$. Divide the interval $[a,b]$ into n equal parts each of length less than δ, by choosing a positive integer n such that $\dfrac{b-a}{n} < \delta$ and real numbers x_k, $k = 0,\ldots,n$, such that

$$a = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n = b,$$

where $x_{k+1} - x_k = \dfrac{b-a}{n}$ for $k = 0,\ldots,n-1$. Also, divide the interval $[-M,M]$ into m equal parts each of length less than $\frac{1}{4}\varepsilon$ by choosing a positive integer m such that $\dfrac{2M}{m} < \frac{1}{4}\varepsilon$ and real numbers y_j, $j = 0,\ldots,m$, such that

$$-M = y_0 < y_1 < \cdots < y_j < y_{j+1} < \cdots < y_m = M$$

where $y_{j+1} - y_j = \dfrac{2M}{m}$ for $j = 0,\ldots,m-1$. In this way we divide the rectangle $[a,b] \times [-M,M]$ into mn sub-rectangles.

We use this subdivision to obtain the finite collection $\mathscr{G}$ of piecewise-linear functions. A typical member g of $\mathscr{G}$ is constructed in the following way: for each $k = 0,\ldots,m$, choose an integer $j(k)$ such that $0 \leq j(k) \leq m$; then the graph of g is made up of the line segments joining $(x_k, y_{j(k)})$ to $(x_{k+1}, y_{j(k+1)})$ for $k = 0,\ldots,n-1$ (see diagram below). There are $(m+1)^{n+1}$ such functions g; these functions g form the finite collection $\mathscr{G}$ of piecewise-linear functions which will serve our purpose.

Let $f \in \mathscr{F}$. It remains to show that there exists g in $\mathscr{G}$ such that $f \in B_\varepsilon(g)$. Note that if $x_k \leq x \leq x_{k+1}$ then $|x - x_k| < \delta$ so that from (1) we have

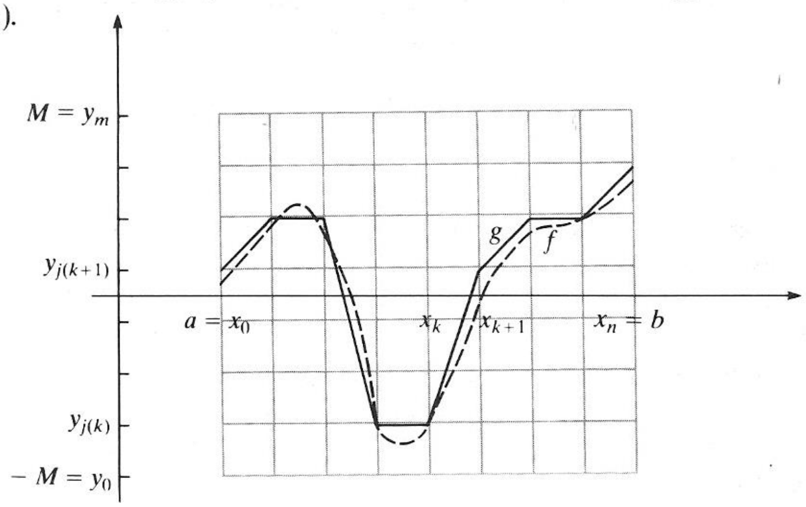
$$|f(x) - f(x_k)| < \tfrac{1}{4}\varepsilon. \qquad (2)$$

In particular, taking $x = x_{k+1}$ gives

$$-\tfrac{1}{4}\varepsilon < f(x_{k+1}) - f(x_k) < \tfrac{1}{4}\varepsilon \qquad (3)$$

for $k = 0,\ldots,n-1$.

We now construct the required member g of $\mathscr{G}$. Of course, since g is piecewise-linear, we need only specify the values of $g(x_k)$ for $k = 0,\ldots,n$. For each $k = 0,\ldots,n$, we take $g(x_k)$ to be the least of the real numbers y_i such that $y_i \geq f(x_k)$.



Note that this choice implies that

$$0 \leq g(x_k) - f(x_k) < \tfrac{1}{4}\varepsilon \qquad (4)$$

for $k = 0,\ldots,n$. Since $f(x_k) \leq g(x_k)$, it follows that

$$\begin{aligned} g(x_{k+1}) - g(x_k) &\leq g(x_{k+1}) - f(x_k) \\ &= \{g(x_{k+1}) - f(x_{k+1})\} + \{f(x_{k+1}) - f(x_k)\}; \end{aligned}$$

so that $g(x_{k+1}) - g(x_k) < \frac{1}{2}\varepsilon$, from (3) and (4). Similarly, since $g(x_{k+1}) \geq f(x_{k+1})$, it follows that

$$\begin{aligned} g(x_{k+1}) - g(x_k) &\geq f(x_{k+1}) - g(x_k) \\ &= \{f(x_{k+1}) - f(x_k)\} + \{f(x_k) - g(x_k)\}; \end{aligned}$$

so that $g(x_{k+1}) - g(x_k) > -\frac{1}{2}\varepsilon$, from (3) and (4). Thus we have

$$|g(x_{k+1}) - g(x_k)| < \tfrac{1}{2}\varepsilon$$

for $k = 0, \ldots, n-1$. Since g is linear on $[x_k, x_{k+1}]$, it follows that, if $x_k \leq x \leq x_{k+1}$, then

$$|g(x) - g(x_k)| \leq |g(x_{k+1}) - g(x_k)| < \tfrac{1}{2}\varepsilon. \qquad (5)$$

Now if $x_k \leq x \leq x_{k+1}$ we have

$$|f(x) - g(x)| \leq |f(x) - f(x_k)| + |f(x_k) - g(x_k)| + |g(x_k) - g(x)|;$$

so that from (2), (4) and (5) we have

$$|f(x) - g(x)| < \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This is true for $k = 0, 1, \ldots, n-1$, and so

$$|f(x) - g(x)| < \varepsilon$$

for all x in $[a,b]$. It follows from Proposition 4 that $f \in B_\varepsilon(g)$ as required. ■

From the preceding proof we can also deduce the following result which tells us that, in a sense, each continuous real-valued function on $[a,b]$ can be approximated arbitrarily closely by piecewise-linear functions. We shall exploit this result in Section 8.6.

ROPOSITION 10 (Piecewise-Linear Approximation Theorem) If $f \in \mathbf{C}[a,b]$ and $\varepsilon > 0$, then there exists a piecewise-linear function g in $\mathbf{C}[a,b]$ such that $d_\infty(f,g) < \varepsilon$.

PROOF Since f is uniformly continuous on $[a,b]$, it follows that the singleton set $\{f\}$ is equicontinuous. Since $\{f\}$ is obviously bounded in $\mathbf{C}[a,b]$, from the proof of Proposition 9 we deduce that there exists a piecewise-linear function g on $[a,b]$ such that $f \in B_\varepsilon(g)$, i.e. such that $d_\infty(f,g) < \varepsilon$. ■

We are now ready for the main result of this section.

ROPOSITION 11 (The Arzelà–Ascoli Theorem) A subset of $\mathbf{C}[a,b]$ is compact if and only if it is bounded, closed and equicontinuous.

PROOF Let $\mathscr{F}$ be a bounded, closed, equicontinuous subset of $\mathbf{C}[a,b]$. Since $\mathbf{C}[a,b]$ is complete (by Example 10), it follows from **S** Proposition 9.2.3(b) that the subspace $\mathscr{F}$ is complete. Also, it follows from Proposition 9 that $\mathscr{F}$ is totally bounded. Thus $\mathscr{F}$ is compact by **S** Theorem 10.1.8.

Conversely, suppose that $\mathscr{F}$ is a compact subspace of $\mathbf{C}[a,b]$. Then, by **S** Propositions 5.4.1 and 5.4.2 (and using the fact that every metric space is a Hausdorff space), $\mathscr{F}$ is closed and bounded in $\mathbf{C}[a,b]$. It remains to show that $\mathscr{F}$ is equicontinuous.

Let $\varepsilon > 0$ be given. We must show that

> there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all f in $\mathscr{F}$
> and all x, y in $[a,b]$ such that $|x - y| < \delta$. $\qquad (*)$

By **S** Propositions 7.2.6 and 7.2.9, $\mathscr{F}$ is totally bounded; and so there is a finite subset $\{f_1, \ldots, f_k\}$ of $\mathscr{F}$ which is a $(\frac{1}{3}\varepsilon)$-net for $\mathscr{F}$. By **S** Proposition 5.8.2, for each $i = 1, \ldots, k$, f_i is uniformly continuous on $[a,b]$, so that there exists $\delta_i > 0$ such that

$$|f_i(x) - f_i(y)| < \tfrac{1}{3}\varepsilon$$

for all x, y in $[a,b]$ such that $|x - y| < \delta_i$. Now let

$$\delta = \operatorname{Min}\{\delta_1, \ldots, \delta_k\}.$$

Then $\delta > 0$ satisfies $(*)$. For if $f \in \mathscr{F}$, then there exists some i, $1 \leq i \leq k$, such that $d_\infty(f, f_i) < \frac{1}{3}\varepsilon$; and so

$$|f(t) - f_i(t)| < \tfrac{1}{3}\varepsilon \qquad \text{for all } t \text{ in } [a, b].$$

Now if $x, y \in [a, b]$ and $|x - y| < \delta$ then $|x - y| < \delta_i$ so that

$$|f_i(x) - f_i(y)| < \tfrac{1}{3}\varepsilon.$$

Hence

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon. \end{aligned}$$

Thus $(*)$ is satisfied. ■

SAQ 20 Prove that the subset $\mathscr{F}$ of $\mathscr{C}[a, b]$ which consists of all functions f which satisfy the condition

$$|f(x) - f(y)| \leq K|x - y|, \qquad x, y \in [a, b],$$

where K is some fixed positive constant, is equicontinuous.

SOLUTION TO SAQ 20 Let $\varepsilon > 0$ be given, and take $\delta = \dfrac{\varepsilon}{K} > 0$. If $f \in \mathscr{F}$ and $x, y \in [a, b]$ with $|x - y| < \delta$, then

$$|f(x) - f(y)| < K\delta = \varepsilon.$$

Thus $\mathscr{F}$ is equicontinuous. ■

8.6 Baire's Theorem for Complete Metric Spaces

Baire's Theorem is a major theorem in topology, and has many applications. We end with one of these, which we hope you will find both surprising and interesting. For the proof of Baire's Theorem, we need the characterization of completeness known as Cantor's Intersection Theorem, with which we begin.

READ S from page 134, line 1, to page 134, line 21.

NOTES

(i) *Page 134, line 5*
We ask you to prove this assertion in SAQ 21.

(ii) *Page 134, lines 8 to 10*
Let us amplify the argument to show that V_n is closed in M.

We argue by contradiction. Suppose that $x \in Cl(V_n)$ and $x \notin V_n$. Then x is a limit point of the set of members of the sequence (x_r) and so, by **S** Lemma 7.2.5, (x_r) has a subsequence which converges to x. But (x_r) is a Cauchy sequence so that, by **S** Proposition 9.2.5, (x_r) converges to x. But this is a contradiction, since (x_r) was assumed to be a non-convergent Cauchy sequence.

Hence $Cl(V_n) = V_n$, and so V_n is closed in M.

(iii) *Page 134, lines 10 to 12*
S proves that given $\varepsilon > 0$, there exists N such that diam $V_N < \varepsilon$. Since $V_n \subset V_N$ for all $n \geq N$, it follows that diam $V_n < \varepsilon$ for all $n \geq N$. Hence diam $V_n \longrightarrow 0$ as $n \longrightarrow \infty$. ▲

SAQ 21 Show that if (V_n) is a sequence of subsets of a metric space such that diam $V_n \longrightarrow 0$ as $n \longrightarrow \infty$, then $\bigcap_{n=1}^{\infty} V_n$ contains at most one point.

SAQ 22 Give examples in $\mathbf{R}$, with the Euclidean metric, of:

(a) a sequence (W_n) of non-empty open sets with $\bigcap_{n=1}^{\infty} W_n = \varnothing$ such that $W_{n+1} \subset W_n$ and diam $W_n \longrightarrow 0$ as $n \longrightarrow \infty$;

(b) a sequence (F_n) of closed sets with $\bigcap_{n=1}^{\infty} F_n = \varnothing$ such that $F_{n+1} \subset F_n$ for every n and F_n is unbounded for every n.

(This question is intended to emphasize just what the conditions of Theorem 9.5.1 are!)

SOLUTION TO SAQ 21 Suppose that a and b are *distinct* points of $\bigcap_{n=1}^{\infty} V_n$. Then $d(a, b) > 0$. Since diam $V_n \longrightarrow 0$ as $n \longrightarrow \infty$, there exists N such that diam $V_n < d(a, b)$ for all $n \geq N$. But $a \in V_N$ since $a \in \bigcap_{n=1}^{\infty} V_n$; and similarly $b \in V_N$. Hence $d(a, b) < d(a, b)$, which is absurd.

This contradiction shows that $\bigcap_{n=1}^{\infty} V_n$ contains at most one point. ■

SOLUTION TO SAQ 22 (a) Let $W_n = \left(0, \frac{1}{n}\right)$ for each n. Then $W_{n+1} \subset W_n$ for every n, and diam $W_n = \frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$. But $\bigcap_{n=1}^{\infty} W_n = \varnothing$ — for it is clear that if $x \leq 0$ then, for all n, $x \notin W_n$; whilst if $x > 0$, we can choose N such that $\frac{1}{N} \leq x$, so that $x \notin W_N$.

(b) Let $F_n = [n, \infty)$ for each n. Then $F_{n+1} \subset F_n$ for every n and each F_n is an unbounded closed set. If $x \in \mathbf{R}$, there exists a positive integer N such that $N > x$. Then $x \notin F_N$ and so $x \notin \bigcap_{n=1}^{\infty} F_n$. Therefore $\bigcap_{n=1}^{\infty} F_n = \varnothing$. ■

READ S from page 134, line 23, to page 135, line −10.

NOTE *Page 135, line −12*
From the construction by induction, we see that for each n

$$V_n = \{y \in M : d(x_n, y) \leq \tfrac{1}{2} r_n\},$$

where $r_n < \frac{1}{n}$. If $x, y \in V_n$, then

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq r_n;$$

so that

$$\text{diam } V_n \leq r_n < \frac{1}{n}.$$

Hence diam $V_n \longrightarrow 0$ as $n \longrightarrow \infty$. ▲

As an example of the use of Baire's Theorem, in which the completeness of the metric space $\mathbf{C}[0, 1]$ is exploited, we are going to show that there exist continuous real-valued functions on $[0, 1]$ which are *not* differentiable at *any* interior point of $[0, 1]$! We recall that if $f \colon [0, 1] \rightarrow \mathbf{R}$ is a function and $0 < x < 1$, then f is said to be *differentiable* at x if

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \text{ exists.}$$

If this limit exists, its value is denoted by $f'(x)$. From the discussion of limits in Section 2.3 of *Unit 2*, we know that if f is differentiable at x, then given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| < \varepsilon$$

for all h such that $0 < |h| < \delta$.

For one step in the proof of our final Proposition, we shall need

LEMMA 12 If the sequence (f_n) converges to g in $\mathbf{C}[0,1]$ and (x_n) converges to x_0 in $[0,1]$, then

$$\lim_{n\to\infty} f_n(x_n) = g(x_0).$$

PROOF Let $\varepsilon > 0$ be given. Since (f_n) converges to g in $\mathbf{C}[0,1]$, there exists a positive integer N_1 such that

$$|g(x) - f_n(x)| < \tfrac{1}{2}\varepsilon$$

for all x in $[0,1]$ and all $n \geq N_1$. In particular, if $n \geq N_1$ then

$$|g(x_n) - f_n(x_n)| < \tfrac{1}{2}\varepsilon.$$

Since g is continuous and $x_n \longrightarrow x_0$ as $n \longrightarrow \infty$, it follows from Proposition 1 that $g(x_n) \longrightarrow g(x_0)$ as $n \longrightarrow \infty$. Thus there exists a positive integer N_2 such that

$$|g(x_0) - g(x_n)| < \tfrac{1}{2}\varepsilon$$

for all $n \geq N_2$.

Now let $N = \text{Max}\{N_1, N_2\}$. Then for all $n \geq N$ we have

$$\begin{aligned} |g(x_0) - f_n(x_n)| &\leq |g(x_0) - g(x_n)| + |g(x_n) - f_n(x_n)| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Therefore $f_n(x_n) \longrightarrow g(x_0)$ as $n \longrightarrow \infty$. ■

We now come to the main result; as we shall find, the proof actually establishes very much more than we state in the Proposition itself. This is discussed in the Remark at the end of the proof.

PROPOSITION 13 There exists a continuous real-valued function on $[0,1]$ which is differentiable at *no* point of $(0,1)$.

PROOF This is long, so we have broken it up into stages.

I Definition of H_n and H

For each positive integer n, let H_n be the subset of $\mathbf{C}[0,1]$ consisting of those functions f which satisfy the following condition: there exists x in $\left[0, 1-\dfrac{1}{n}\right]$ such that for all h in $\left(0, \dfrac{1}{n}\right)$ we have

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leq n.$$

Put $H = \bigcup_{n=1}^{\infty} H_n$.

II Each function differentiable at some point of $(0,1)$ *belongs to H*

For, suppose that f is differentiable at a where $0 < a < 1$. Then, taking $\varepsilon = 1$ in the definition of differentiability, we see that there exists $\delta > 0$ such that

$$\left|\frac{f(a+h)-f(a)}{h} - f'(a)\right| < 1$$

for all h such that $0 < |h| < \delta$; so that, if $0 < |h| < \delta$, then (by the Triangle Inequality)

$$\left|\frac{f(a+h)-f(a)}{h}\right| \le \left|\frac{f(a+h)-f(a)}{h} - f'(a)\right| + |f'(a)|$$
$$< 1 + |f'(a)|.$$

If we can choose a positive integer N such that

(i) $a \in \left[0, 1 - \frac{1}{N}\right]$,

(ii) $\frac{1}{N} \le \delta$, and

(iii) $N \ge 1 + |f'(a)|$,

then it will follow that $f \in H_N$. Thus we choose an integer N such that

$$N \ge \mathrm{Max}\left\{\frac{1}{1-a}, \frac{1}{\delta}, 1 + |f'(a)|\right\}.$$

Since $N \ge \frac{1}{1-a}$ and $a > 0$ we see that $a \in \left[0, 1 - \frac{1}{N}\right]$.

Also, since $N \ge \frac{1}{\delta}$, we see that if $h \in \left(0, \frac{1}{N}\right)$ then $0 < h < \delta$. Thus $a \in \left[0, \frac{1}{N}\right]$ and for all h in $\left(0, \frac{1}{N}\right)$ we have

$$\left|\frac{f(a+h)-f(a)}{h}\right| \le 1 + |f'(a)| \le N.$$

Therefore $f \in H_N \subset H$.

III Strategy of the proof

It follows from II that if

$$\mathbf{C}[0,1] - H \ne \varnothing$$

then there exists a continuous function on $[0,1]$ which is not differentiable at any point of $(0,1)$: each member of $\mathbf{C}[0,1] - H$ has this property.

So we apply Baire's Theorem to the complete metric space $\mathbf{C}[0,1]$. If we can show that each H_n is nowhere-dense in $\mathbf{C}[0,1]$, then it will follow from Baire's Theorem that $\mathbf{C}[0,1] - H$ is everywhere-dense in $\mathbf{C}[0,1]$, and so certainly is non-empty (as required).

IV H_n is closed

To show that each H_n is closed in $\mathbf{C}[0,1]$, let $g \in Cl(H_n)$. We must show that $g \in H_n$. By **S** Lemma 7.2.2, there exists a sequence (f_i) in H_n which converges to g in $\mathbf{C}[0,1]$. Since $f_i \in H_n$, for each i there exists x_i in $\left[0, 1 - \frac{1}{n}\right]$ such that

$$\left|\frac{f_i(x_i+h)-f_i(x_i)}{h}\right| \le n \qquad (1)$$

for all h in $\left(0, \frac{1}{n}\right)$. Now (x_i) is a sequence in the compact set $\left[0, 1 - \frac{1}{n}\right]$; and so, by **S** Proposition 7.2.6, (x_i) has a subsequence, $(x_{i(k)})$ say, which converges to a point, x_0 say, of $\left[0, 1 - \frac{1}{n}\right]$. Now the sequence $(f_{i(k)})$ converges to g in $\mathbf{C}[0,1]$. It follows from Lemma 12 that

$$f_{i(k)}(x_{i(k)}) \longrightarrow g(x_0) \text{ as } k \longrightarrow \infty.$$

Also, if $h \in \left(0, \frac{1}{n}\right)$ then $x_{i(k)} + h \longrightarrow x_0 + h$ as $k \longrightarrow \infty$, so that

$$f_{i(k)}(x_{i(k)} + h) \longrightarrow g(x_0 + h) \text{ as } k \longrightarrow \infty.$$

Therefore, for all h in $\left(0, \frac{1}{n}\right)$,

$$\left|\frac{f_{i(k)}(x_{i(k)} + h) - f_{i(k)}(x_{i(k)})}{h}\right| \longrightarrow \left|\frac{g(x_0 + h) - g(x_0)}{h}\right|$$

as $k \longrightarrow \infty$. It follows, by letting $i \longrightarrow \infty$ in (1) through the subsequence $(i(k))$, that

$$\left|\frac{g(x_0 + h) - g(x_0)}{h}\right| \leq n$$

for all h in $\left(0, \frac{1}{n}\right)$. Thus there exists x_0 in $\left[0, 1 - \frac{1}{n}\right]$ such that for all h in $\left(0, \frac{1}{n}\right)$ we have

$$\left|\frac{g(x_0 + h) - g(x_0)}{h}\right| \leq n.$$

Hence $g \in H_n$. Therefore H_n is closed in $\mathbf{C}[0,1]$.

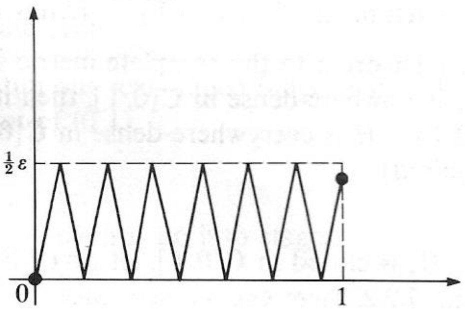
V *H_n is nowhere-dense in* $\mathbf{C}[0,1]$

To establish that each H_n is nowhere-dense in $\mathbf{C}[0,1]$, since each H_n is closed it is sufficient to show that $\text{Int}(H_n) = \varnothing$ for each n.

Let $f \in H_n$ and let $\varepsilon > 0$ be given. We must show that $B_\varepsilon(f)$ is not contained in H_n. Thus we must find g in $\mathbf{C}[0,1] - H_n$ such that $d_\infty(f, g) < \varepsilon$.

By Proposition 10, there exists a piecewise-linear function G in $\mathbf{C}[0,1]$ such that $d_\infty(f, G) < \frac{1}{2}\varepsilon$. The graph of G consists of a finite number of line-segments; let M be the maximum of the absolute values of the slopes of these line-segments. Then for each x in $\left[0, 1 - \frac{1}{n}\right]$ and for all sufficiently small h in $\left(0, \frac{1}{n}\right)$ we have

$$\left|\frac{G(x + h) - G(x)}{h}\right| \leq M. \qquad (2)$$



Next we construct a 'saw-tooth function' ϕ (see the diagram) such that $0 \leq \phi(x) \leq \frac{1}{2}\varepsilon$ for all x in $[0,1]$ and the absolute value of the slope of each of the line-segments which make up its graph is equal to $M + n + 1$. Notice that if $x \in \left[0, 1 - \frac{1}{n}\right]$, then for all sufficiently small h in $\left(0, \frac{1}{n}\right)$ we have

$$\left|\frac{\phi(x + h) - \phi(x)}{h}\right| = M + n + 1. \qquad (3)$$

Now let $g = G + \phi$. We shall show that g is the required member of $\mathbf{C}[0,1] - H_n$. Note that for all x in $[0,1]$,

$$\begin{aligned} |f(x) - g(x)| &= |f(x) - G(x) - \phi(x)| \\ &\leq |f(x) - G(x)| + |\phi(x)| \\ &< \varepsilon; \end{aligned}$$

so that $d_\infty(f, g) < \varepsilon$.

It remains to show that $g \notin H_n$. Let $x \in \left[0, 1 - \frac{1}{n}\right]$; we must consider

$$\left|\frac{g(x+h) - g(x)}{h}\right| = \left|\left(\frac{G(x+h) - G(x)}{h}\right) - \left(\frac{\phi(x) - \phi(x+h)}{h}\right)\right|$$
$$\geq \left|\left|\frac{G(x+h) - G(x)}{h}\right| - \left|\frac{\phi(x+h) - \phi(x)}{h}\right|\right|,$$

using the 'cunning form' of the triangle inequality. There exists h in $\left(0, \frac{1}{n}\right)$ such that (2) and (3) are satisfied. For such h we have

$$\left|\frac{G(x+h) - G(x)}{h}\right| - \left|\frac{\phi(x+h) - \phi(x)}{h}\right| \leq M - (M + n + 1)$$
$$= -(n+1),$$

so that

$$\left|\left|\frac{G(x+h) - G(x)}{h}\right| - \left|\frac{\phi(x+h) - \phi(x)}{h}\right|\right| \geq n + 1.$$

Thus for each x in $\left[0, 1 - \frac{1}{n}\right]$ there exists h in $\left(0, \frac{1}{n}\right)$ such that

$$\left|\frac{g(x+h) - g(x)}{h}\right| \geq n + 1 > n.$$

Therefore $g \notin H_n$.

This completes the proof that H_n has empty interior, and so each H_n is nowhere-dense in $\mathbf{C}[0,1]$. ■

Remark. Let $\mathscr{E}$ denote the set of continuous real-valued functions on $[0,1]$ which are differentiable at no point of $(0,1)$. In the preceding proof we have shown that

$$\mathbf{C}[0,1] - H \subset \mathscr{E};$$

and $\mathbf{C}[0,1] - H$ is everywhere-dense in $\mathbf{C}[0,1]$. Hence $\mathscr{E}$ is everywhere-dense in $\mathbf{C}[0,1]$.

Thus we can say that, in a sense, almost all continuous real-valued functions on $[0,1]$ are differentiable at *no* point of $(0,1)$!

SAQ 23 Prove that if, for each positive integer n, U_n is a dense open set in a complete metric space M, then $\bigcap_{n=1}^{\infty} U_n$ is dense in M. [Hint: Put $H_n = M - U_n$ for each n.]

SAQ 24 Prove that if M is a complete metric space and $M = \bigcup_{n=1}^{\infty} E_n$ where each E_n is closed in M, then at least one of the sets E_n has non-empty interior.

SOLUTION TO SAQ 23 Put $H_n = M - U_n$ for each n. Then H_n is closed in M for each n. If $x \in H_n$ and V is an open set of M such that $x \in V$, then $V \cap U_n \neq \varnothing$ since $x \in Cl(U_n) = M$. Thus V is not contained in H_n. It follows that the interior of H_n is empty. Hence, for each n, H_n is nowhere-dense in M.

It follows from Baire's Theorem that $M - \bigcup_{n=1}^{\infty} H_n$ is everywhere-dense in M. But by Proposition 3 of *Unit 1*, page 16, we have

$$M - \bigcup_{n=1}^{\infty} H_n = \bigcap_{n=1}^{\infty} (M - H_n) = \bigcap_{n=1}^{\infty} U_n;$$

hence $\bigcap_{n=1}^{\infty} U_n$ is dense in M. ■

SOLUTION TO SAQ 24 It follows from **S** Definition 3.7.26 that a closed set is nowhere-dense if and only if it has empty interior. If the interior of E_n were empty for each n, then, by Baire's Theorem, $M - \bigcup_{n=1}^{\infty} E_n$ would be dense in M and so certainly non-empty. Since $M = \bigcup_{n=1}^{\infty} E_n$, it follows that at least one of the sets E_n has non-empty interior. ■

8.7 Objectives of Unit 8

After studying this unit, you should know:

(1) the definition of a Cauchy sequence in a metric space, that each convergent sequence is a Cauchy sequence, and that in some metric spaces there are non-convergent Cauchy sequences;

(2) that a subset C of a metric space is compact if and only if each sequence in C contains a subsequence which converges to some point of C;

(3) the definition of a Lebesgue number of an open cover of a compact metric space;

(4) the definition of total boundedness;

(5) the main properties of the metric spaces $\mathbf{B}[a,b]$ and $\mathbf{C}[a,b]$;

(6) the definition of completeness, the simple properties of complete spaces, and that $\mathbf{R}^n$, $\mathbf{B}[a,b]$ and $\mathbf{C}[a,b]$ are complete metric spaces;

(7) that a metric space is compact if and only if it is complete and totally bounded;

(8) the characterization of compactness of a subset of $\mathbf{C}[a,b]$;

(9) the properties of a complete metric space formulated in the Cantor Intersection Theorem and in Baire's Theorem;

(10) that there exists a function continuous on $[0,1]$ differentiable at no point of $(0,1)$.

8.8 Revision Questions

SAQ 25 For each positive integer n, let f_n: $[0,1] \to \mathbf{R}$ be defined by

$$f_n(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 - \dfrac{1}{n}, \\ n(1-x), & \text{if } 1 - \dfrac{1}{n} \le x \le 1. \end{cases}$$

(a) Prove that each f_n is continuous.

(b) Show that (f_n) is not a Cauchy sequence in $\mathbf{C}[0,1]$.

***SAQ 26** Let $M = \{\mathbf{R}, e\}$ where e is the metric on $\mathbf{R}$ (see Example 1 of *Unit 3*, page 69) given by

$$e(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \text{Max}\{|x|, |y|\}, & \text{if } x \ne y. \end{cases}$$

(a) Show that every sequence which converges to zero in $\mathbf{R}$, with the Euclidean metric, is a Cauchy sequence in M.

(b) The eventually-constant sequences in M are Cauchy sequences in M (as you saw in the Remark on page 84). Show that a non-eventually constant

sequence is a Cauchy sequence in M if and only if it converges to 0 in **R** with the Euclidean metric.
(Do not spend more than 10 minutes on this part, before looking at its solution.)

(c) Deduce that M is a complete metric space.

SAQ 27 Let M be a compact metric space, and suppose that $(F_i)_{i\in I}$ is a family of closed sets of M such that $\bigcap_{i\in I} F_i = \varnothing$. Prove that there exists a positive number ε with the following property: if C is a subset of M with diam $C < \varepsilon$ then there exists some $j \in I$ such that $C \cap F_j = \varnothing$. (In other words, if C is sufficiently small, there is at least one of the Fs that it does not meet.)

[Hint: The family $(M - F_i)_{i\in I}$ is an open cover of the compact space M, and so has a Lebesgue number ε.]

SAQ 28 Compare **R**, with the Euclidean metric, with its subspace $(0, 1)$ relative to compactness, completeness and total boundedness. (Recall that these two spaces are homeomorphic.)

SAQ 29 Prove that a subset of a complete metric space is compact if and only if it is closed and totally bounded.

SAQ 30 (a) Prove that if M is a countably infinite complete metric space, then there exists a point a in M such that $\{a\}$ is open in M. [Hint: Apply the result of SAQ 24 to an enumeration of M.]

(b) Deduce that there is no metric d on **Q**, topologically equivalent to the Euclidean metric, such that $\{\mathbf{Q}, d\}$ is a complete metric space. [Hint: Use a contradiction argument.]

SOLUTION TO SAQ 25

(a) This is an immediate consequence of Proposition 1 of *Unit 7*, for $f_n \bigg| \left[0, 1 - \frac{1}{n}\right]$ and $f_n \bigg| \left[1 - \frac{1}{n}, 1\right]$ are evidently continuous.

(b) We argue by contradiction. Suppose that (f_n) *is* a Cauchy sequence in $\mathbf{C}[0, 1]$. Since $\mathbf{C}[0, 1]$ is a complete metric space, it follows that (f_n) converges to f in $\mathbf{C}[0, 1]$. Then, as shown in Section 8.3, we have

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{for all } x \text{ in } [0, 1].$$

But if $0 \leq x < 1$ and N is a positive integer such that $1 - \frac{1}{N} \geq x$ $\left(\text{i.e. } N \geq \frac{1}{1-x}\right)$, then $0 \leq x \leq 1 - \frac{1}{N} \leq 1 - \frac{1}{n}$ for all $n \geq N$ and so $f_n(x) = 1$ for all $n \geq N$. Thus $f(x) = 1$ if $0 \leq x < 1$. Also, $f_n(1) = 0$ for every n, so that $f(1) = 0$. Thus $f\colon [0, 1] \to \mathbf{R}$ is given by

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

Since it is evident that f is not continuous, we have a contradiction. Thus (f_n) is not a Cauchy sequence in $\mathbf{C}[0, 1]$. ■

SOLUTION TO SAQ 26

(a) Suppose that (x_n) is a sequence which converges to 0 in **R** with the Euclidean metric. Let $\varepsilon > 0$ be given.

We must show that there exists a positive integer N such that $e(x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

Since (x_n) converges to 0 in **R** with the Euclidean metric, there exists a positive integer N such that $|x_n| < \varepsilon$ for all $n \geq N$. Thus, if $m, n \geq N$, then

$$e(x_n, x_m) \leq \text{Max}\{|x_n|, |x_m|\} < \varepsilon.$$

Therefore (x_n) is a Cauchy sequence in M.

(b) We have seen in part (a) that each sequence which converges to 0 in **R** with the Euclidean metric is a Cauchy sequence in M.

Now let (x_n) be a sequence which does not converge to 0 in **R** with the Euclidean metric and which is not eventually constant. We shall show that (x_n) is not a Cauchy sequence in M, by a contradiction argument.

Suppose that (x_n) *is* a Cauchy sequence in M. Since (x_n) does not converge to 0 in **R** with the Euclidean metric, there exists $\varepsilon_0 > 0$ such that $|x_n| \geq \varepsilon_0$ for infinitely many values of n. Since (x_n) is a Cauchy sequence in M, there exists a positive integer N such that $e(x_n, x_m) < \varepsilon_0$ for all m, $n \geq N$. Now there exists $j > N$ such that $|x_j| \geq \varepsilon_0$. Also, since (x_n) is not eventually constant, there exists $k > N$ such that $x_j \neq x_k$. Then j, $k > N$, and

$$e(x_j, x_k) = \text{Max}\{|x_j|, |x_k|\} \geq |x_j| \geq \varepsilon_0.$$

This contradiction shows that (x_n) is not a Cauchy sequence in M.

(c) We must show that each Cauchy sequence in M is convergent.

Of course, each sequence which is eventually constant is convergent in M.

Next, let (x_n) be a sequence in **R** which converges to 0 in **R** with the Euclidean metric. Then given $\varepsilon > 0$, there exists N such that $|x_n| < \varepsilon$ for all $n \geq N$. But as noted in Example 10 of *Unit 3*, page 82,

$$B_\varepsilon(0; e) = (-\varepsilon, \varepsilon);$$

so that $x_n \in B_\varepsilon(0; e)$ for all $n \geq N$. Hence (x_n) converges to 0 in M.

Thus every Cauchy sequence in M is convergent, and so M is a complete metric space. ■

SOLUTION TO SAQ 27

Since $\bigcap_{i\in I} F_i = \varnothing$, it follows from Proposition 3 of *Unit 1*, page 16, that

$$\bigcup_{i\in I} (M - F_i) = M - \left(\bigcap_{i\in I} F_i\right) = M.$$

Also, since F_i is closed in M, $M - F_i$ is open in M for each i. Therefore $(M - F_i)_{i\in I}$ is an open cover of M. Since M is a compact metric space, by **S** Lemma 7.2.12 this open cover has a Lebesgue number, ε say. Thus for each x in M, there exists some j in I such that $B_\varepsilon(x) \subset M - F_j$. Now let C be a non-empty subset of M with diam $C < \varepsilon$. If $c \in C$ there exists some j in I such that $B_\varepsilon(c) \subset M - F_j$. But since diam $C < \varepsilon$, it follows that $C \subset B_\varepsilon(c) \subset M - F_j$, and so $C \cap F_j = \varnothing$. ■

SOLUTION TO SAQ 28

R with the Euclidean metric is complete, not compact and hence not totally bounded. The subspace $(0, 1)$ is totally bounded, not compact and not complete. ■

SOLUTION TO SAQ 29

We know from Note (iv) of Section 8.5 that a metric space is compact if and only if it is complete and totally bounded.

Thus if K is a compact subset of a complete metric space M then K is complete and totally bounded. Also, since K is complete, K is closed in M by **S** Proposition 9.2.3(a).

Conversely, if K is closed in a complete metric space M then K is complete by **S** Proposition 9.2.3(b). Thus if K is closed and totally bounded in a complete metric space, then K is complete and totally bounded; and hence is compact. ■

OLUTION TO SAQ 30

(a) If M is a countably infinite complete metric space and its points are enumerated as $\{a_1, a_2, \ldots\}$, then $M = \bigcup_{n=1}^{\infty} \{a_n\}$; and, since M is a metric space, each singleton set $\{a_n\}$ is closed. It follows from SAQ 24 (which is an immediate consequence of Baire's Theorem) that, for some n, $\{a_n\}$ has non-empty interior. In other words, for that n, $\{a_n\}$ must be open.

(b) We argue by contradiction. Suppose that there exists some metric d on $\mathbf{Q}$, topologically equivalent to the Euclidean metric, such that $\{\mathbf{Q}, d\}$ is complete. Then, from part (a), there exists some q in $\mathbf{Q}$ such that $\{q\}$ is open in $\{\mathbf{Q}, d\}$. Since d is topologically equivalent to the Euclidean metric on $\mathbf{Q}$, it follows that $\{q\}$ is open in $\mathbf{Q}$ with the Euclidean metric. But this is a contradiction; for by the rational density property (of *Unit 1*), each open ball in $\mathbf{Q}$ with the Euclidean metric contains infinitely many points. Hence no such metric d exists. ■

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